RESEARCH STATEMENT

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Non-Specialist Summary

My first main research project is centered around a nearly 40-year-old open problem by Lê Dũng Tráng concerning the equisingularity of germs of complex analytic surfaces $V(f) := \{f = 0\}$ in $\mathbb{C}^3$ with one-dimensional singular locus. Lê's Conjecture concerns the deep relationship between the topological and analytic properties of surface germs: if the link of $V(f)$ (the intersection of $V(f)$ with a sufficiently small sphere centered at 0) is homeomorphic to a 3-sphere, then the singular locus of $V(f)$ is a smooth curve at 0. The link of a surface is one of the most important pieces of data associated to a singularity, and this hypothesis places strict constraints on the local topology of $V(f)$ at 0–in particular, it implies that the normalization of $V(f)$ is smooth, and is a bijection. One pictures $V(f)$ as "folding up" $\mathbb{C}^2$, with the "creases" corresponding to the singular locus. My approach, together with Laurentiu Maxim, is via the machinery of Saito's mixed Hodge modules.

My second research project centers around topological aspects of the Riemann-Hilbert correspondence for holonomic D-modules with possibly irregular singularities. For any holomorphic function $f\colon X \to \mathbb{C}$ on a complex manifold X, we define and study moderate growth and rapid decay objects associated to an enhanced ind-sheaf on X. These will be sheaves on the real oriented blow-up space of X along f. We show that in the context of the Riemann–Hilbert functor due to D'Agnolo–Kashiwara, these objects recover the classical de Rham complexes with moderate growth and rapid decay associated to a holonomic D-module. Moreover, we resolve a conjectural duality of Sabbah between these de Rham complexes in the normal crossing case, and recover in particular a well-known duality pairing for integrable connections on smooth varieties.

1. Research Summary: Lê's Conjecture

This project is the investigation of a classic conjecture (due to Lê [198]) in the field of singularities of complex analytic spaces, specifically on the so-called "equisingularity" of certain surfaces with non-isolated singularities inside $\mathbb{C}^3$. Let us briefly recall some of the essential notions from singularity theory.

The study of (the topology of) complex hypersurfaces with **isolated** singularities largely started with the foundational work of Milnor [Mil68]; the local, ambient topological type of a hypersurface $V(f) \subseteq \mathbb{C}^{n+1}$ at a singular point $p \in V(f)$ is completely determined by a fibration (called the Milnor fibration) defined on a "tube" around the hypersurface near p. More precisely, for $0 < \delta \ll \epsilon \ll 1$, the defining function f restricts to a smooth, locally trivial fibration

$$\hat{f} : B_\epsilon(p) \cap f^{-1}(\partial \mathbb{D}_\delta) \to \partial \mathbb{D}_\delta$$

where $B_\epsilon(p)$ is an open ball of radius ϵ at p in $\mathbb{C}^{n+1}$ (with respect to any Riemannian metric), and $\mathbb{D}_\delta$ is a disk of radius δ around 0 in $\mathbb{C}$. The fiber of $\hat{f}$ is called the **Milnor fiber** of

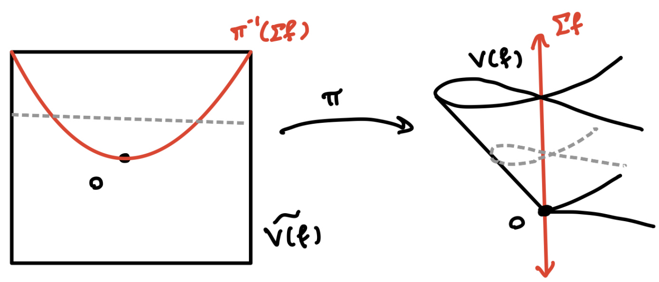
f at p, denoted $F_{f,p}$, is a compact, orientable manifold of dimension $2n$ that is homotopy equivalent to a finite bouquet of n-spheres. The number of such spheres is called the **Milnor number** of f at p, and is denoted $\mu_p(f)$.

Milnor's fibration still exists in the more general context of hypersurfaces with non-isolated singularities, but the associated Milnor fiber is no longer as nice as in the isolated case. It still has the homotopy-type of a finite CW complex, and if the singular locus Σf has $\dim_0 \Sigma f = s$, then a classical result of Kato-Matsumoto [Kat73] tells us that $\widetilde{H}^k(F_{f,p}; \mathbb{Z}) \neq 0$ only for $n - s \leq k \leq n$, and the Milnor fiber is contractible if and only if p is a non-singular point of $V(f)$. The majority of the study of non-isolated hypersurface singularities boils down to understanding these cohomology groups.

The next "easiest" case to examine after isolated singularities is, of course, hypersurfaces with one-dimensional singularities. Here, we know that $F_{f,p}$ can only have non-trivial cohomology in degrees n and $n-1$, and still it is highly non-trivial to compute these groups in general. The general setting of Lê's conjecture is then interesting primarily because, for surfaces in $\mathbb{C}^3$, there is not "enough room" for complicated topological phenomena to happen, even with non-isolated singularities. We now state the precise conjecture:

Problem 1.1 (Lê's Conjecture [198])**.** *Let $(V(f), \mathbf{0}) \subseteq (\mathbb{C}^3, 0)$ be a reduced complex analytic surface with $\dim_0 \Sigma f = 1$, such that the normalization $\pi : (\widetilde{V(f)}, 0) \to (V(f), 0)$ is smooth, and π is a bijection. Then, $V(f)$ is isomorphic to the total space of an equisingular deformation of an irreducible plane curve singularity.*

Since the problem is local, we may assume $\widetilde{V(f)}$ is just $\mathbb{C}^2$. Additionally, we will not precisely define the general notion of "equisingular deformation" here, but it suffices to say that there exists a generic linear function $L : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ such that $V(L)$ transversely intersects Σf at 0, and for all $t \in \mathbb{C}$ small, the Milnor number $\mu_0(f_{|_{V(L-t)}})$ is independent of t. In this way, we can regard $V(f)$ as a family of plane curves $V(f, L-t)$ with isolated singularities that are all "the same".



Non-example: the normalization of the Whitney umbrella $y^2 = x^3 + zx^2$ is smooth, but not a bijection. Hence it is **not** an equisingular deformation of the cusp $y^2 = x^3$.

Despite this Conjecture having been around for nearly 40 years, it is only known to be true in a handful of special cases: when Σf contains a smooth curve, when $V(f)$ is a cyclic cover of a normal surface singularity, and when f is a sum of two homogeneous forms, to name a few (see e.g., [Bob06],[Bob06] for a complete list of known cases). It is suspected that perhaps new theory must be developed to attack this problem, or that it may involve more of the interplay between the analytic and topological properties of surface germs.

My approach to this problem is centered around the technical machinery of perverse sheaves and mixed Hodge modules. The bulk of my previous work (and Ph.D. thesis!) concerned the study of non-isolated hypersurface singularities with smooth normalizations via perverse sheaves [Hep16],[Hep18],[Hep19a],[Hep19b]. In particular, we recover Bobadilla's result as a special case of the main results of [Hep18]. Via the language of perverse sheaves, it is then possible to rephrase the Conjecture in terms of the complex of vanishing cycles $\varphi_f[-1]\mathbb{Q}^\bullet_{\mathbb{C}^3}[3]$. We will not go into the details of the language of perverse sheaves, or the derived category here for the sake of brevity, but loosely $\varphi_f[-1]\mathbb{Q}^\bullet_{\mathbb{C}^3}[3]$ is a complex of sheaves of finite dimensional $\mathbb{Q}$-vector spaces whose support is $V(f) \cap \Sigma f$, and for all $p \in \Sigma f$, there is a canonical isomorphism

$$H^k(\varphi_f[-1]\mathbb{Q}^\bullet_{\mathbb{C}^3}[3])_p \cong \widetilde{H}^{2+k}(F_{f,p};\mathbb{Q})$$

There is a natural monodromy action T_f on the cohomology of the Milnor fiber, given by allowing the values of f to travel in a circle around the origin in $\mathbb{C}$. This extends to the level of the derived category to a natural isomorphism of perverse sheaves (also denoted T_f) on $\varphi_f[-1]\mathbb{Q}^\bullet_{\mathbb{C}^3}[3]$. The Milnor monodromy operator has a Jordan decomposition $T_f = T_f^u \circ T_f^s$ where T_f^u is unipotent, and T_f^s is semi-simple of finite order. For $\lambda \in \mathbb{Q}$, the (generalized) eigenspaces $\varphi_{f,\lambda} := \ker\{T_f^s - \lambda \cdot Id\}$ are perverse subsheaves of $\varphi_f[-1]\mathbb{Q}^\bullet_{\mathbb{C}^3}[3]$, and there is a natural splitting $\varphi_f[-1]\mathbb{Q}^\bullet_{\mathbb{C}^3}[3] \cong \varphi_{f,1} \oplus \varphi_{f,\neq 1}$. Via my work in [Hep19b], the hypotheses of Lê's Conjecture imply the vanishing $\varphi_{f,1} = 0$; consequently, the non-unipotent vanishing cycles become the central object of study.

Perverse sheaves represent the topological perspective of my approach to proving this Conjecture; to see the analytic structure we have mentioned, we "enhance" these objects to mixed Hodge modules. These are, broadly speaking, perverse sheaves whose stalks are all mixed Hodge structures, and generically are variations of (polarizable) mixed Hodge structures. Together with Laurentiu Maxim at the University of Wisconsin-Madison (where I was previously employed as a postdoc), we have reduced Lê's Conjecture to the following statement.

Problem 1.2 (H., Maxim)**.** *Let $V(f)$ be as in Problem 1.1. Then, the non-unipotent vanishing cycles $\varphi_{f,\neq 1}$ is a semi-simple mixed Hodge module that is pure of weight 2.*

Intuitively, semi-simplicity removes the "obstruction" to $V(f)$ being an equisingular deformation.

2. Irregular Singularities

The theories of ind-sheaves (initiated in [Kas01]) and enhanced ind-sheaves (established in [D'A16]) led to an extension of the classical Riemann–Hilbert correspondence for regular holonomic D-modules, which had been proved by M. Kashiwara in [Kas84] (see also Z. Mebkhout [Meb84] for a different proof) and which states that the de Rham functor

$$\mathrm{DR}_X\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{reg,hol}}(\mathcal{D}_X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$$

from the derived category of regular holonomic D-modules to the derived category of $\mathbb{C}$-constructible sheaves on a complex manifold X is an equivalence of categories.

It was not difficult to observe that this functor is no longer fully faithful on the category of (not necessarily regular) holonomic $\mathcal{D}_X$-modules, and finding an irregular analogue for the target category was a long-standing problem that led to the development of the above-mentioned theories and the following result:

On a complex manifold X, A. D'Agnolo and M. Kashiwara (see [D'A16]) defined a functor

$$\mathrm{DR}_X^{\mathrm{E}} : \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X) \hookrightarrow \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X),$$

called the *enhanced de Rham functor*, from the derived category of holonomic D-modules to the category of so-called *enhanced ind-sheaves*. More precisely, the essential image of this functor is contained in the subcategory of $\mathbb{R}$-constructible enhanced ind-sheaves.[1]

Classical objects in the study of differential equations with irregular singularities are the de Rham complexes with moderate growth and rapid decay. The reason why such complexes are important is the following: There are non-isomorphic D-modules whose (classical) de Rham complexes are isomorphic. Roughly speaking, this happens because their sheaves of holomorphic solutions are the same, even though the growth behavior of their solutions is different, but the classical de Rham functor is not sensitive to growth conditions. In order to obtain a functor which can distinguish such D-modules, it is therefore necessary to introduce variants of the de Rham functor that can "measure" the growth of the solutions to a differential system. A technical difficulty in these constructions is the need to work on real oriented blow-up spaces, where functions with moderate growth and rapid decay are well-defined. Such blow-ups naturally arise as the Kato-Nakayama spaces in logarithmic geometry.

The simple idea at the heart of this project is the following: Since $\mathrm{DR}_X^{\mathrm{E}}$ is fully faithful, all the information about a holonomic D-module must be encoded in its enhanced de Rham complex. In particular, there should be a functorial way to obtain the moderate growth and rapid decay de Rham complexes of a holonomic D-module $\mathcal{M}$ from the enhanced ind-sheaf $\mathrm{DR}_X^{\mathrm{E}}(\mathcal{M})$, without leaving the topological setting. The aim of my paper [Hep22] with Andreas Hohl was to develop an analogous theory of nearby and vanishing cycles for enhanced ind-sheaves that is compatible with the irregular Riemann–Hilbert correspondence, i.e., a theory that produces the nearby cycles defined in [Sab21] if applied to the enhanced de Rham complex of a holonomic $\mathcal{D}_X$-module on an arbitrary complex manifold X.

In particular, in order to to this, we investigate in [Hep22] two notions that do not seem to have been studied in the context of enhanced ind-sheaves yet: Firstly, we develop the notion of enhanced de Rham complex on the real blow-up along a holomorphic function f (in [D'A16] and [Kas16], only the real blow-up along a normal crossing divisor has been studied). Secondly, we study the notion of rapid decay function and the rapid decay de Rham complex using enhanced ind-sheaves. (In [D'A16] and [Kas16], only the sheaf $\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}}$ of holomorphic functions with moderate growth has been studied). In order to achieve results about rapid decay objects on this different notion of real blow-up, we mimic constructions done in [D'A16] in the case of moderate growth along a normal crossing divisor, and then make a connection between moderate growth and rapid decay by duality, using a duality result of M. Kashiwara and P. Schapira (see [Kas16]) about the closely related notions of tempered and Whitney functions. In this way, we proved—in the case of a normal crossing divisor—a conjecture posed by C. Sabbah in [Sab21].

[1] In fact, one can restrict the target category further: The essential image is precisely the category of $\mathbb{C}$-constructible enhanced ind-sheaves, a notion introduced in [Ito20] (see also [Kuw21] for an alternative approach). In this way, the functor $\mathrm{DR}_X^{\mathrm{E}}$ becomes an equivalence of categories, but we will not go into these notions here.

Theorem 1 ([Hep22])**.** *Let $D \subset X$ be a simple normal crossing divisor, and let $\widetilde{X}$ be the real oriented blow-up of X along D. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Then there is an isomorphism in $D^b(\mathbb{C}_{\widetilde{X}})$*

$$D_{\widetilde{X}}\, \mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}}(\mathcal{M}) \cong \mathrm{DR}^{\mathrm{rd}}_{\widetilde{X}}(\mathbb{D}_X\mathcal{M})$$

where $D_{\widetilde{X}}$ denotes the Verdier dual functor in $D^b(\mathbb{C}_{\widetilde{X}})$, and $\mathbb{D}_X$ denotes the dual functor in $D^b_{hol}(\mathcal{D}_X)$.

A special case of this theorem is the well-known period pairing of Bloch-Esnault and Hien:

Corollary 1 ([Hep22])**.** *Let (E, ∇) be a flat (algebraic) connection on a smooth quasi-projective variety U over $\mathbb{C}$, and let $(E^\vee, \nabla^\vee)$ be the dual connection on U. Then there is a perfect pairing of finite-dimensional $\mathbb{C}$-vector spaces*

$$\mathrm{H}^\ell_{\mathrm{dR}}\big(U; (E,\nabla)\big) \otimes \mathrm{H}^{\mathrm{rd}}_\ell\big(U; (E^\vee, \nabla^\vee)\big) \to \mathbb{C}$$

where H_{dR} denotes algebraic de Rham cohomology, and H^{rd} denotes rapid decay homology (see [Blo04] and [Hie09]).

In future work, we hope to relax the normal crossings assumption on the divisor, and further develop the many interesting connections with log D-modules.

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