MODERATE AND RAPID DECAY NEARBY CYCLES VIA ENHANCED IND-SHEAVES

BRIAN HEPLER AND ANDREAS HOHL

Abstract. For any holomorphic function $f\colon X \to \mathbb{C}$ on a complex manifold X, we define and study moderate and rapid decay objects associated to an enhanced ind-sheaf on X. These will be sheaves on the real oriented blow-up space of X along f. We show that in the context of the Riemann–Hilbert functor due to D'Agnolo–Kashiwara, these objects recover the classical de Rham complexes with moderate growth and rapid decay associated to a holonomic D-module. Moreover, we resolve a conjectural duality of Sabbah between these de Rham complexes in the normal crossing case, and recover in particular a well-known duality pairing for integrable connections on smooth varieties.

Contents

Date: June 2022.

The research of A.H. is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Research Fellowship HO 6925/1-1.

1. Introduction

The theories of ind-sheaves (initiated in [KS01]) and enhanced ind-sheaves (established in [DK16]) led to an extension of the classical Riemann–Hilbert correspondence for regular holonomic D-modules, which had been proved independently by M. Kashiwara [Kas84] and Z. Mebkhout [Meb84]) and which states that the de Rham functor

$$\mathrm{DR}_X\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{reg,hol}}(\mathcal{D}_X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$$

from the derived category of regular holonomic D-modules to the derived category of $\mathbb{C}$-constructible sheaves on a complex manifold X is an equivalence of categories.

It was not difficult to observe that this functor is no longer fully faithful on the category of (not necessarily regular) holonomic $\mathcal{D}_X$-modules, and finding an irregular analogue for the target category was a long-standing problem that led to the development of the above-mentioned theories and the following result:

On a complex manifold X, A. D'Agnolo and M. Kashiwara (see [DK16]) defined a functor

$$\mathrm{DR}^{\mathrm{E}}_X : \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X) \hookrightarrow \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X),$$

called the *enhanced de Rham functor*, from the derived category of holonomic D-modules to the category of so-called *enhanced ind-sheaves*. More precisely, the essential image of this functor is contained in the subcategory of $\mathbb{R}$-constructible enhanced ind-sheaves.[1]

From a general point of view, this result teaches us that constructible enhanced ind-sheaves should be our objects of interest in the study of irregular singularities—just as we were interested in constructible sheaves as the topological counterparts of regular holonomic D-modules. More precisely, it will be helpful to have tools like a theory of nearby and vanishing cycles at hand in the context of these still quite recent objects. One of the purposes of this paper is therefore to introduce objects similar to the ones known from the classical theory of constructible sheaves.

Classical objects in the study of differential equations with irregular singularities are the de Rham complexes with moderate growth and rapid decay. The reason why such complexes are important is the following: There are non-isomorphic D-modules whose (classical) de Rham complexes are isomorphic. Roughly speaking, this happens because their sheaves of holomorphic solutions are the same, even though the growth behavior of their solutions is different, but the classical de Rham functor is not sensitive to growth conditions. In order to obtain a functor which can distinguish such D-modules, it is therefore necessary to introduce variants of the de Rham functor that can "measure" the growth of the solutions to a differential system. A technical difficulty in these constructions is the need to work on real oriented blow-up spaces, where functions with moderate growth and rapid decay are well-defined. (Let us note that the use of the real blow-up can, however, partly be avoided by working with ind-sheaves.)

In the one-dimensional case, a solution to the irregular Riemann–Hilbert problem has been described earlier by introducing the notion of *Stokes structure*, which can be formulated in different ways, one of them being *Stokes-filtered local systems*

[1]In fact, one can restrict the target category further: The essential image is precisely the category of $\mathbb{C}$-constructible enhanced ind-sheaves, a notion introduced in [Ito20] (see also [Kuw21] for an alternative approach). In this way, the functor $\mathrm{DR}^{\mathrm{E}}_X$ becomes an equivalence of categories, but we will not use these notions here.

(see [Mal91], and we refer to [Sab13] for an exposition on Stokes filtrations for D-modules, and to [Boa21] for a history and comparison of different formulations of the Stokes phenomenon). Indeed, the Stokes filtration on the local system associated to a holonomic D-module $\mathcal{M}$ is essentially given by the moderate de Rham complexes of "exponentially twisted" versions of $\mathcal{M}$.

The simple idea at the heart of this article is now the following: Since $\mathrm{DR}_X^{\mathrm{E}}$ is fully faithful, all the information about a holonomic D-module must be encoded in its enhanced de Rham complex. In particular, there should be a functorial way to obtain the moderate growth and rapid decay de Rham complexes of a holonomic D-module $\mathcal{M}$ from the enhanced ind-sheaf $\mathrm{DR}_X^{\mathrm{E}}(\mathcal{M})$, without leaving the topological setting.

Indeed, such a connection between enhanced de Rham complexes and Stokes filtrations has been described in the one-dimensional case in [DK20] (see also [DK18]). In the recent article [Sab21], nearby and vanishing cycles for general holonomic D-modules were defined and studied.

The aim of this paper is now to develop an analogous theory of nearby and vanishing cycles for enhanced ind-sheaves that is compatible with the irregular Riemann–Hilbert correspondence, i.e., a theory that produces the nearby cycles defined in [Sab21] if applied to the enhanced de Rham complex of a holonomic $\mathcal{D}_X$-module on an arbitrary complex manifold X.

In particular, in order to to this, we will investigate two notions that do not seem to have been studied in the context of enhanced ind-sheaves yet: Firstly, we develop the notion of enhanced de Rham complex on the real blow-up along a holomorphic function f (in [DK16] and [KS16], only the real blow-up along a normal crossing divisor has been studied). Secondly, we study the notion of rapid decay function and the rapid decay de Rham complex using enhanced ind-sheaves. (In [DK16] and [KS16], only the sheaf $\mathcal{A}_{\widetilde{X}}^{\mathrm{mod}}$ of holomorphic functions with moderate growth has been studied). In order to achieve results about rapid decay objects on this different notion of real blow-up, we will on the one hand mimick constructions done in [DK16] in the case of moderate growth along a normal crossing divisor, and on the other hand make a connection between moderate growth and rapid decay by duality, using a duality result of M. Kashiwara and P. Schapira (see [KS96] and [KS16]) about the closely related notions of tempered and Whitney functions. In this way, we prove—in the case of a normal crossing divisor—a conjecture posed by C. Sabbah in [Sab21].

Outline of the paper. Section 2 is a brief review of the languages of sheaves, ind-sheaves, enhanced ind-sheaves and D-modules, as well as the theory of real oriented blow-up spaces, all of which we will use in this paper.

In Section 3, we motivate our later definition of moderate and rapid decay objects associated to enhanced ind-sheaves: First, we show how the moderate de Rham complex of a holonomic D-module $\mathcal{M}$ along a divisor can be recovered in a functorial way from its enhanced de Rham complex $\mathrm{DR}_X^{\mathrm{E}}(\mathcal{M})$ (Proposition 3.1). This is greatly inspired by the formulas developed in [DK20], which are briefly reviewed at the beginning of the section. The proof in the higher-dimensional case works in two steps: We prove the result in the case of a simple normal crossing divisor, and then use results about resolution of singularities to prove a statement in the general case. We then motivate a similar functorial construction for the rapid decay de Rham complex of $\mathcal{M}$, which will be proved in the case of normal crossing divisor in Section 6.

In Section 4, we define functors that associate to an enhanced ind-sheaf moderate and rapid decay objects in the category of sheaves on the real blow-up. These naturally lead to definitions for moderate and rapid decay nearby cycles in the category of sheaves on the boundary of the real blow-up space. The definitions will be motivated by the statements in the preceding section, so that these objects will in particular recover moderate de Rham complexes of holonomic D-modules. We also prove some elementary properties of these objects, similar to those known from classical nearby cycles (see [KS90]) and moderate and rapid decay objects associated to holonomic D-modules (as in [Sab21]).

If one starts with an $\mathbb{R}$-constructible enhanced ind-sheaf, the objects defined above are easily seen to be related by duality, and hence one has natural duality pairings betweens them that we study in Section 5.

In Section 6, after reviewing a duality result of [KS96] and [KS16] between tempered and Whitney functions, we prove a conjecture of [Sab21] in the case of a simple normal crossing divisor and show that duality interchanges the moderate growth with rapid decay de Rham complex for holonomic D-modules. This result will have two interesting implications: First of all, it shows that the rapid decay object defined in Section 4 does indeed recover the rapid decay de Rham complex of a holonomic D-module (Corollary 6.11). Moreover, it allows us to recover the pairings between rapid decay homology and de Rham cohomology due to M. Hien [Hie09] (Proposition 6.12).

Acknowledgments. First of all, we would like to thank Claude Sabbah who pointed out many questions studied in this article to us, and in particular inspired us with his recent work on nearby cycles for holonomic D-modules in the higher-dimensional case. We thank him and Andrea D'Agnolo for useful discussions and correspondence during the preparation of this work. Moreover, we are grateful to Pierre Schapira for explanations on the duality between Whitney and tempered functions, which helped us in proving the duality in Section 6.

2. Background and notation

2.1. **From sheaves to enhanced ind-sheaves.**

In this section, we will recall some basic notation in the context of sheaf theory and its generalizations. Let X be a topological space (all topological spaces will be assumed to be *good*, i.e., Hausdorff, locally compact, second countable and of finite flabby dimension), and let k be a field.

Sheaves. We denote by $\mathrm{Mod}(k_X)$ the category of sheaves of k-vector spaces on X and by $\mathrm{D^b}(k_X)$ its bounded derived category. One has the six Grothendieck operations $\mathrm{R}\mathcal{H}om, \otimes, \mathrm{R}f_*, f^{-1}, \mathrm{R}f_!, f^!$ (for f a morphism of good topological spaces) on $\mathrm{D^b}(k_X)$. We denote the duality functor by $\mathrm{D}_X = \mathrm{R}\mathcal{H}om_{k_X}(\bullet, \omega_X)$, where ω_X is the dualizing complex. For a locally closed set $Z \subseteq X$, we will denote by $k_Z \in \mathrm{Mod}(k_X)$ the constant sheaf with stalk k on Z, extended by zero outside Z. We refer to the standard literature, e.g., [KS90], for details on sheaf theory.

Ind-sheaves. We denote by $\mathrm{I}(k_X)$ the category of ind-sheaves over k on X, which has been constructed in [KS01] as the category of ind-objects for the category of compactly supported sheaves of k-vector spaces. The inductive limit in $\mathrm{I}(k_X)$ will be denoted by "$\varinjlim$". We denote by $\mathrm{D}^{\mathrm{b}}(\mathrm{I}k_X)$ the bounded derived category of $\mathrm{I}(k_X)$. One has the six Grothendieck operations $\mathrm{R}\mathcal{I}hom, \otimes, \mathrm{R}f_*, f^{-1}, \mathrm{R}f_{!!}, f^!$ on it. Moreover, one has a fully faithful embedding $\iota_X \colon \mathrm{Mod}(k_X) \to \mathrm{I}(k_X)$. It has a left adjoint α_X, which itself has a fully faithful left adjoint β_X. Let us note that the three functors ι_X, α_X and β_X are exact. Let us also note that the natural inclusion ι_X is mostly suppressed in the notation (starting with [KS01]) and that also the functor β_X is often suppressed in the more recent notational conventions of [DK16] and [KS16] (and works thereafter). We refer to [KS01] for details on the theory of ind-sheaves.

Enhanced ind-sheaves. The category of enhanced ind-sheaves has been constructed in [DK16] in order to establish a Riemann–Hilbert correspondence for holonomic D-modules.

To this end, the authors of loc. cit. introduced the notion of a *bordered space*, which is a pair $(X, \widehat{X})$ of good topological spaces such that $X \subseteq \widehat{X}$ is an open subspace. A morphism of bordered spaces $(X, \widehat{X}) \to (Y, \widehat{Y})$ is a continuous map $X \to Y$ such that, denoting by $\overline{\Gamma}$ the closure of its graph in $\widehat{X} \times \widehat{Y}$, the map $\overline{\Gamma} \to \widehat{X}$ induced by the projection to the first factor is proper. (Note that this condition is in particular satisfied if $\widehat{Y}$ is compact.) The category of good topological spaces is naturally a subcategory of the category of bordered spaces by considering a topological space X as the pair (X, X).

Let $\mathcal{X} = (X, \widehat{X})$ be a bordered space, and consider also the bordered space $\mathbb{R}_\infty := (\mathbb{R}, \mathsf{P})$, where $\mathsf{P} := \mathbb{P}^1(\mathbb{R})$ is the real projective line. One then defines the quotient categories

$$\mathrm{D}^{\mathrm{b}}(\mathrm{I}k_{\mathcal{X}\times\mathbb{R}_\infty}) := \mathrm{D}^{\mathrm{b}}(\mathrm{I}k_{\widehat{X}\times\mathsf{P}})/\mathrm{D}^{\mathrm{b}}(\mathrm{I}k_{(\widehat{X}\times\mathsf{P})\setminus(X\times\mathbb{R})})$$

and

$$\mathrm{E}^{\mathrm{b}}(\mathrm{I}k_{\mathcal{X}}) := \mathrm{D}^{\mathrm{b}}(\mathrm{I}k_{\mathcal{X}\times\mathbb{R}_\infty})/\pi^{-1}\mathrm{D}^{\mathrm{b}}(\mathrm{I}k_{\mathcal{X}}),$$

where $\pi = \pi_{\mathcal{X}} \colon \mathcal{X} \times \mathbb{R}_\infty \to \mathcal{X}$ denotes the projection. We refer to [DK16] for details on this construction and to [KS06] for an exposition on quotients of triangulated categories. One calls $\mathrm{E}^{\mathrm{b}}(\mathrm{I}k_{\mathcal{X}})$ the category of *enhanced ind-sheaves* over k on $\mathcal{X}$. It admits the six Grothendieck operations $\mathrm{R}\mathcal{I}hom^+, \overset{+}{\otimes}, \mathrm{E}f_*, \mathrm{E}f^{-1}, \mathrm{E}f_{!!}$ and $\mathrm{E}f^!$ for a morphism of bordered spaces f. Moreover, we denote by $\mathrm{D}^{\mathrm{E}}_{\mathcal{X}}$ the duality functor for enhanced ind-sheaves.

A basic object in the category $\mathrm{E}^{\mathrm{b}}(\mathrm{I}k_{\mathcal{X}})$ is

$$k^{\mathrm{E}}_{\mathcal{X}} := \text{“}\varinjlim_{a\to\infty}\text{”}\, k_{\{t\geq a\}},$$

where we abbreviate $\{t \geq a\} := \{(x, t) \in \widehat{X} \times \mathsf{P}; x \in X, t \in \mathbb{R}, t \geq a\}$.

For a bordered space $\mathcal{X} = (X, \widehat{X})$, there is a fully faithful embedding

$$\begin{aligned} e_{\mathcal{X}} \colon \mathrm{D}^{\mathrm{b}}(k_X) &\hookrightarrow \mathrm{E}^{\mathrm{b}}(\mathrm{I}k_{\mathcal{X}}), \\ F &\mapsto k^{\mathrm{E}}_{\mathcal{X}} \otimes \pi^{-1}F. \end{aligned}$$

In [DK21b], the authors introduced the *sheafification functor* $\mathsf{sh}_{\mathcal{X}} \colon \mathrm{E}^{\mathrm{b}}(\mathrm{I}k_{\mathcal{X}}) \to \mathrm{D}^{\mathrm{b}}(k_X)$ for enhanced ind-sheaves on a bordered space $\mathcal{X} = (X, \widehat{X})$. In the case of stable (in particular for $\mathbb{R}$-constructible, see below) enhanced ind-sheaves, it is given by

$$\mathsf{sh}_{\mathcal{X}}(K) := \mathrm{R}\mathcal{H}om^{\mathrm{E}}(k_{\mathcal{X}}^{\mathrm{E}}, K) \in \mathrm{D}^{\mathrm{b}}(k_X).$$

(For a definition of the functor $\mathrm{R}\mathcal{H}om^{\mathrm{E}}$, we refer to [DK16, Definition 4.5.13], where it was denoted by $\mathcal{H}om^{\mathrm{E}}$. See also [DK21b] for a detailed study of the sheafification functor.) Let us note that $\mathsf{sh}_{\mathcal{X}} \circ e_{\mathcal{X}} \simeq \mathrm{id}_{\mathrm{D}^{\mathrm{b}}(k_X)}$, i.e., $\mathsf{sh}_{\mathcal{X}}$ is a left quasi-inverse of $e_{\mathcal{X}}$.

Constructibility. In classical sheaf theory, different constructibility conditions for sheaves of vector spaces have been studied (see, e.g., [KS90]). In particular, if X is a real analytic manifold (or more generally, a *subanalytic space*, see, e.g., [KS90, Exercise 9.2]), there are the full subcategories $\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X) \subset \mathrm{Mod}(k_X)$ of $\mathbb{R}$-constructible sheaves and $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X) \subset \mathrm{D}^{\mathrm{b}}(k_X)$ of complexes with $\mathbb{R}$-constructible cohomologies.

If X is a complex manifold, a stronger notion is that of a $\mathbb{C}$-constructible sheaf, and one has the full subcategories $\mathrm{Mod}_{\mathbb{C}\text{-c}}(k_X) \subset \mathrm{Mod}(k_X)$ and $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X) \subset \mathrm{D}^{\mathrm{b}}(k_X)$.

For enhanced ind-sheaves on a real analytic bordered space $\mathcal{X} = (X, \widehat{X})$ (meaning that X and $\widehat{X}$ are real analytic manifolds), there is the full subcategory $\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}k_{\mathcal{X}}) \subset \mathrm{E}^{\mathrm{b}}(\mathrm{I}k_{\mathcal{X}})$ (see [DK16, §4.9] for details).

A notion of $\mathbb{C}$-constructibility has been studied in [Ito20] (see also [Kuw21] and [Moc18] for different approaches to describe the essential image of the Riemann–Hilbert functor of [DK16]).

2.2. **D-modules.** When X is a complex manifold, we will denote by $\mathcal{D}_X$ the sheaf of linear partial differential operators with holomorphic coefficients on X. We refer to the standard literature, such as [Kas03, HTT08], for the theory of D-modules (on complex manifolds as well as on smooth algebraic varieties).

Let X be a complex manifold. We denote by $\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$ the category of holonomic $\mathcal{D}_X$-modules and by $\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ the full subcategory of the derived category of $\mathcal{D}_X$-modules consisting of complexes with holonomic cohomologies. We denote by $\mathbb{D}_X$ the duality functor for $\mathcal{D}_X$-modules.

The (classical) de Rham functor is given by

$$\begin{aligned}\mathrm{DR}_X \colon \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X) &\longrightarrow \mathrm{D}^{\mathrm{b}}(\mathbb{C}_X),\\ \mathcal{M} &\longmapsto \Omega_X \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X} \mathcal{M},\end{aligned}$$

where Ω_X is the sheaf of top-degree holomorphic differential forms on X.

In [DK16], A. D'Agnolo and M. Kashiwara defined the *enhanced de Rham functor*

$$\begin{aligned}\mathrm{DR}^{\mathrm{E}}_X \colon \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X) &\longrightarrow \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X),\\ \mathcal{M} &\longmapsto \Omega^{\mathrm{E}}_X \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X} \mathcal{M},\end{aligned}$$

where $\Omega^{\mathrm{E}}_X = \Omega_X \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_X} \mathcal{O}^{\mathrm{E}}_X$, and $\mathcal{O}^{\mathrm{E}}_X$ is the enhanced ind-sheaf of enhanced tempered holomorphic functions, and stated the following Riemann–Hilbert correspondence for holonomic $\mathcal{D}_X$-modules.

Theorem 2.1 (cf. [DK16, Theorem 9.5.3])**.** *The functor* $\mathrm{DR}^{\mathrm{E}}_X$ *is fully faithful.*

The functors e_X and sh_X make a connection between the classical and the enhanced de Rham functor: For a regular holonomic D-module $\mathcal{R} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{reghol}}(\mathcal{D}_X)$, one has

$$\mathrm{DR}^{\mathrm{E}}_X(\mathcal{M}) \simeq e_X \mathrm{DR}_X(\mathcal{M}).$$

On the other hand, for any holonomic D-module $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$, one has

$$\mathsf{sh}_X \mathrm{DR}^{\mathrm{E}}_X(\mathcal{M}) \simeq \mathrm{DR}_X(\mathcal{M}).$$

2.3. **Real blow-up spaces.** In the following, we will recall two (in general different) constructions of a real oriented blow-up space (often simply called *real blow-up*) associated to a complex manifold and a divisor. We will also recall some important sheaves on these blow-up spaces. For more details, we refer to [Sab13], [Moc14], [DK16], [KS16].

Real blow-up along a function. Let X be a complex manifold and $f\colon X \to \mathbb{C}$ a holomorphic function. Then the *real blow-up of X along f* is denoted by $\varpi_f\colon \widetilde{X}_f \to X$ and defined as follows: Consider the map $f/|f|\colon X \to S^1$, then $\widetilde{X}_f$ is the closure of its graph in $X \times S^1$. The map $\varpi_f\colon \widetilde{X}_f \to X$ is induced by the projection to the first factor. It is a homeomorphism on $X^* := X \setminus f^{-1}(0) \simeq \widetilde{X}_f \setminus \partial\widetilde{X}_f$. Moreover, we have $\partial\widetilde{X}_f \simeq f^{-1}(0) \times S^1$.

We denote by X^*_∞ the bordered space $(X^*, X) \simeq (X^*, \widetilde{X}_f)$, and we fix the following notation for the morphisms (all of them inclusions except for ϖ_f) that will appear throughout the paper:

$$\partial\widetilde{X}_f \xhookrightarrow{\ i_f\ } \widetilde{X}_f \xhookleftarrow{\ \tilde{j}_f\ } X^*_\infty \xhookrightarrow{\ j\ } X, \qquad \varpi_f\colon \widetilde{X}_f \to X \tag{1}$$

This construction is functorial in the following sense: Given two complex manifolds X and Y with holomorphic functions $f\colon X \to \mathbb{C}$ and $g\colon Y \to \mathbb{C}$, as well as a morphism of complex manifolds $\tau\colon X \to Y$ such that $g \circ \tau = f$, then there is an induced morphism $\widetilde{\tau}\colon \widetilde{X}_f \to \widetilde{Y}_g$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X}_f & \xrightarrow{\ \widetilde{\tau}\ } & \widetilde{Y}_g \\ {\scriptstyle \varpi_f}\big\downarrow & & \big\downarrow{\scriptstyle \varpi_g} \\ X & \xrightarrow{\ \tau\ } & Y \end{array}$$

In particular, for $X = \mathbb{C}$ and $f(z) = z$, this construction gives the real blow-up space $\widetilde{\mathbb{C}}_0 = \mathbb{R}_{\geq 0} \times S^1$ with $\varpi_0\colon \widetilde{\mathbb{C}}_0 \to \mathbb{C}$ given by $\varpi_0(\rho, e^{i\theta}) = \rho e^{i\theta}$.

For arbitrary X and holomorphic $f\colon X \to \mathbb{C}$, one can then alternatively describe the construction of $\widetilde{X}_f$ as the fiber product $\widetilde{X}_f := X \times_{\mathbb{C}} \widetilde{C}_0$, i.e., the space fitting into the Cartesian diagram

$$\begin{array}{ccc} \widetilde{X}_f & \longrightarrow & \widetilde{\mathbb{C}}_0 \\ {\scriptstyle \varpi_f}\downarrow & \square & \downarrow{\scriptstyle \varpi_0} \\ X & \xrightarrow[f]{} & \mathbb{C} \end{array}$$

Real blow-up along a simple normal crossing divisor. Let X be a complex manifold and $D \subset X$ a normal crossing divisor with smooth components. (We will always mean such a divisor when we say "normal crossing divisor".) Locally, one can write $D = \{z_1 \cdot \ldots \cdot z_r = 0\} = D_1 \cup \ldots \cup D_r$ for an appropriate coordinate system $z_1, \ldots, z_n$ and $r \leq n$, where we write $D_i := \{z_i = 0\}$. Then, setting $f_i(z_1, \ldots, z_n) = z_i$, we can define the real blow-up spaces $\varpi_{f_i} \colon \widetilde{X}_{f_i} \to X$ as above. (This amounts to replacing the coordinates $z_1, \ldots, z_r$ by polar coordinates $\rho_1, e^{i\theta_1}, \ldots, \rho_r, e^{i\theta_r}$, where we allow $\rho_i = 0$.) The *real blow-up space of X along D* is then denoted by $\varpi_D \colon \widetilde{X}_D \to X$ and defined as the fiber product $\widetilde{X}_{f_1} \times_X \ldots \times_X \widetilde{X}_{f_r}$. This map is a homeomorphism on $X^* := X \setminus D \simeq \widetilde{X}_D \setminus \partial \widetilde{X}_D$.

We fix the following notation for the morphisms (all of them inclusions except for ϖ_D) that will appear throughout the paper:

$$\partial \widetilde{X}_D \xhookrightarrow{i_D} \widetilde{X}_D \xhookleftarrow{\tilde{j}_D} (X \setminus D)_\infty \xhookrightarrow{j} X, \qquad \varpi_D \colon \widetilde{X}_D \to X \tag{2}$$

In the context of a normal crossing divisor, one can also define the *total real blow-up* $\widetilde{X}_D^{\mathrm{tot}}$ (which amounts to allowing $\rho_i \in \mathbb{R}$ in the construction above). It contains the real blow-up $\widetilde{X}_D$ as a closed subset, but—contrarily to the latter—is a real analytic manifold, which will be useful in some situations. Note, however, that it is not globally intrinsically defined (see [DK16, Remark 7.1.1] and [KS16, Remark 4.2.1])

Comparison between the two constructions. While the second definition requires the divisor to have normal crossings and smooth components, the first definition works in a more general setting. In the case of a normal crossing divisor given as the zero set of a holomorphic function, we therefore have two different notions of real blow-up spaces: Let D be a normal crossing divisor given by $D = f^{-1}(0)$ for some holomorphic function $f \colon X \to \mathbb{C}$, then we can define the spaces $\widetilde{X}_D$ and $\widetilde{X}_f$, and there is a natural proper morphism $\varpi_{D,f} \colon \widetilde{X}_D \to \widetilde{X}_f$ and a commutative diagram

$$\begin{array}{ccc} \widetilde{X}_D & \xrightarrow{\varpi_{D,f}} & \widetilde{X}_f \\ & {\scriptstyle \varpi_D}\searrow \quad \swarrow{\scriptstyle \varpi_f} & \\ & X & \end{array}$$

It is easy to see that in the example $X = \mathbb{C}$, $f(z) = z$, the two constructions of real blow-ups coincide.

Sheaves of holomorphic functions with moderate growth and rapid decay. Let $\widetilde{X}$ denote any of the real blow-up spaces defined above. Then one has the sheaves $\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}}$ and $\mathcal{A}^{\mathrm{rd}}_{\widetilde{X}}$ of functions on $\widetilde{X}$ which are holomorphic in the interior X^* and have moderate growth and rapid decay at the boundary $\partial\widetilde{X}$, respectively.

For details on these notions, see, e.g., [Sab13, §8.3], [Moc14, §4.1.2, §4.1.5], [KS01, §7.2], [DK16, §5.1].

Remark 2.2. It is worth noting that the definitions of moderate (or polynomial) growth are phrased slightly differently in different works such as [KS01], [Sab13] and [Moc14]. However, they all lead to the same sheaf of holomorphic functions with moderate growth at the boundary.

Our notation $\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}}$ is closest to that in [Sab13] and [Moc14]. This sheaf is denoted by $\mathcal{A}_{\widetilde{X}}$ in [DK16]. however, it is not the same as the sheaf $\mathcal{A}_{\widetilde{X}}$ in [Sab00], and also should not be confused with the sheaf $\mathscr{A}_{\widetilde{X}}$ in [Sab13] or the sheaf $\mathcal{A}$ in [Mal91].

In fact, it is not completely obvious from the beginning that the sheaf $\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}}$, as defined above, is the same as the sheaf $\mathcal{A}_{\widetilde{X}}$ defined in [DK16, Notation 7.2.1]: In loc. cit., it is defined as the sheaf of functions that are holomorphic in the interior of $\widetilde{X}$ and tempered at the boundary $\partial\widetilde{X}$. This is a priori a stronger condition than the one imposed above (and in [Sab13], [Moc14], for example), since *tempered* means that the function *and all its derivatives* are of moderate growth. For holomorphic functions with moderate growth, this is, however, automatic due to Cauchy's integral formula for derivatives, as is shown in [Siu70, Lemma 3].

De Rham complexes on real blow-ups. Let X be a complex manifold and let a normal crossing divisor $D \subset X$ or a holomorphic function $f\colon X \to \mathbb{C}$ be given, and denote by $\widetilde{X}$ any of the real blow-ups $\widetilde{X}_D$ or $\widetilde{X}_f$.

Definition 2.3. *The* moderate de Rham complex *of a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is defined as (see, e.g.,* [Sab13][2]*)*

$$\begin{aligned}\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}}(\mathcal{M}) &:= (\varpi^{-1}\Omega_X \otimes_{\varpi^{-1}\mathcal{O}_X} \mathcal{A}^{\mathrm{mod}}_{\widetilde{X}}) \overset{\mathrm{L}}{\otimes}_{\varpi^{-1}\mathcal{D}_X} \varpi^{-1}\mathcal{M}\\ &\simeq \varpi^{-1}\Omega_X \overset{\mathrm{L}}{\otimes}_{\varpi^{-1}\mathcal{D}_X} (\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}} \otimes_{\varpi^{-1}\mathcal{O}_X} \varpi^{-1}\mathcal{M}).\end{aligned}$$

We set $\mathcal{M}^{\mathcal{A}} := \mathcal{A}^{\mathrm{mod}}_{\widetilde{X}} \otimes_{\varpi^{-1}\mathcal{O}_X} \varpi^{-1}\mathcal{M}$.

Similarly, the rapid decay de Rham complex *is defined as*

$$\begin{aligned}\mathrm{DR}^{\mathrm{rd}}_{\widetilde{X}}(\mathcal{M}) &:= (\varpi^{-1}\Omega_X \otimes_{\varpi^{-1}\mathcal{O}_X} \mathcal{A}^{\mathrm{rd}}_{\widetilde{X}}) \overset{\mathrm{L}}{\otimes}_{\varpi^{-1}\mathcal{D}_X} \varpi^{-1}\mathcal{M}\\ &\simeq \varpi^{-1}\Omega_X \overset{\mathrm{L}}{\otimes}_{\varpi^{-1}\mathcal{D}_X} (\mathcal{A}^{\mathrm{rd}}_{\widetilde{X}} \otimes_{\varpi^{-1}\mathcal{O}_X} \varpi^{-1}\mathcal{M}).\end{aligned}$$

[2] We remark that we follow here the convention of [Kas03]: The objects DR_X, $\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}}$ etc. here are those denoted by ${}^p\mathrm{DR}_X$, ${}^p\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}}$ in [Sab13], i.e., these objects are already shifted appropriately such that, for instance, $\mathrm{DR}_X(\mathcal{M})$ is a perverse sheaf.

3. De Rham complexes with growth conditions and enhanced ind-sheaves

In this section, we recall and put together some notions and constructions from previous works on enhanced ind-sheaves. The main aim is to motivate the formulas for our definitions in the following sections, by making the connection to the moderate and rapid decay de Rham functors.

3.1. **Dimension one.** Let us first recall the construction of nearby cycles in dimension one given by A. D'Agnolo and M. Kashiwara in [DK20].

Let X be a complex analytic curve and $\mathcal{M}$ a holonomic $\mathcal{D}_X$-module. Let $a \in X$ be a point and denote by $\varpi\colon \widetilde{X} \to X$ the real oriented blow-up of X at a. Set $S_aX := \varpi^{-1}(a) = \partial\widetilde{X}$ and note that $X \setminus \{a\} \simeq \widetilde{X} \setminus S_aX$.

Consider the morphisms

$$S_aX \xhookrightarrow{\ i\ } \widetilde{X} \xhookleftarrow{\ \tilde{j}\ } (X \setminus \{a\})_\infty \xhookrightarrow{\ j\ } X$$

where $(X \setminus \{a\})_\infty$ is the bordered space $(X \setminus \{a\}, X) \simeq (X \setminus \{a\}, \widetilde{X})$.

In [DK20] (see also [DK18, §6]), the authors define the sheaves

$$\begin{aligned} \Psi_a^{\leq 0}(K) &:= i^{-1}\mathsf{sh}_{\widetilde{X}}(\mathrm{E}\tilde{j}_*\mathrm{E}j^{-1}K)[-1] \in \mathrm{D^b}(\mathbb{C}_{S_aX}), \\ \Psi_a^{0}(K) &:= \mathsf{sh}_{S_aX}(\mathrm{E}i^{-1}\mathrm{E}\tilde{j}_*\mathrm{E}j^{-1}K)[-1] \in \mathrm{D^b}(\mathbb{C}_{S_aX}) \end{aligned} \tag{3}$$

associated to an $\mathbb{R}$-constructible enhanced ind-sheaf $K \in \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_X)$.

Moreover, if $I \subset S_aX$ is open and f is a function in a neighbourhood $U \subset \widetilde{X}$ of I which is holomorphic on $U \setminus S_aX$ and admits a Puiseux series expansion on I, they define

$$\begin{aligned} \Psi_a^{\leq f}(K) &:= \Psi_a^{\leq 0}(K(f))|_I \in \mathrm{D^b}(\mathbb{C}_I), \\ \Psi_a^{f}(K) &:= \Psi_a^{0}(K(f))|_I \in \mathrm{D^b}(\mathbb{C}_I), \end{aligned}$$

where $K(f) := \mathrm{R}\mathcal{I}hom^+(\mathbb{E}_{U|X}^{\mathrm{Re}\, f}, K)$. They prove that for $K = \mathrm{DR}_X^{\mathrm{E}}(\mathcal{M})$ these formulas recover the Stokes filtration of $\mathcal{M}$ on the local system of flat sections $\mathcal{L}_{\mathcal{M}} := i^{-1}\tilde{j}_*j^{-1}\mathrm{DR}_X(\mathcal{M})[-1]$ on S_aX. In particular, $\Psi^{\leq 0}(K)$ is the moderate growth de Rham complex of $\mathcal{M}$ on $\widetilde{X}$ (restricted to S_aX) and $\Psi^0(K)$ is the graded piece of the Stokes filtration corresponding to the regular part of the Levelt–Turrittin decomposition.

3.2. **Moderate de Rham complexes in higher dimensions: normal crossing divisors.** We will now show how the ideas of [DK20] naturally extend to the higher-dimensional case for $\mathcal{D}_X$-modules on a manifold with a (simple) normal crossing divisor.

Let X be complex manifold and $D \subset X$ a simple normal crossing divisor. Consider the real oriented blow-up along the components of D and recall the notation for morphisms from (2).

As the next proposition shows, a formula similar to (3) recovers the moderate de Rham complex from the enhanced de Rham complex of a holonomic D-module in the normal crossing case. Its proof is probably not new to experts and relies heavily on results already established in [DK16], but we give it here since it does not seem to appear in other works. In particular, it gives a direct proof for the fact that the object (3) recovers the moderate growth part of the Stokes filtration in the one-dimensional case.

Proposition 3.1. *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module and $D \subset X$ a simple normal crossing divisor. Then there is an isomorphism in* $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\widetilde{X}_D})$

$$\mathsf{sh}_{\widetilde{X}_D}\big(\mathrm{E}\tilde{j}_{D*}\mathrm{E}j^{-1}\mathrm{DR}_X^{\mathrm{E}}(\mathcal{M})\big) \simeq \mathrm{DR}_{\widetilde{X}_D}^{\mathrm{mod}}(\mathcal{M}).$$

In particular, if $\dim_{\mathbb{C}} X = 1$*, we have*

$$\Psi_a^{\leq 0}\big(\mathrm{DR}_X^{\mathrm{E}}(\mathcal{M})\big) \simeq F_{\leq 0}\mathcal{L}_{\mathcal{M}},$$

where the right-hand side denotes the "≤ 0" part of the Stokes filtration on the local system $\mathcal{L}_{\mathcal{M}} := i_D^{-1}\mathrm{R}\tilde{j}_{D*}j^{-1}\mathrm{DR}_X(\mathcal{M})$.

In the proof of this proposition, we will make use of the following lemma. It is the analogue for the de Rham functor on the real blow-up of [DK16, Lemma 9.7.1] (which shows that $\mathsf{sh}_X\mathrm{DR}_X^{\mathrm{E}}(\mathcal{M}) \simeq \mathrm{DR}_X(\mathcal{M})$, although the sheafification functor was not denoted like this in loc. cit.). We will use the constructions and notation for objects on the real blow-up along a normal crossing divisor from [DK16, §7, §9.2].

Lemma 3.2. *Let* $\mathcal{N} \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}^{\mathcal{A}}_{\widetilde{X}_D})$*. There is an isomorphism in* $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\widetilde{X}_D})$

$$\mathsf{sh}_{\widetilde{X}_D}\mathrm{DR}_{\widetilde{X}_D}^{\mathrm{E}}(\mathcal{N}) \simeq \mathrm{DR}_{\widetilde{X}_D}(\mathcal{N}).$$

Proof. Let us abbreviate $\widetilde{X} := \widetilde{X}_D$ and $\varpi := \varpi_D$. First, one has isomorphisms

$$\begin{aligned}\mathsf{sh}_{\widetilde{X}}\mathrm{DR}_{\widetilde{X}}^{\mathrm{E}}(\mathcal{N}) &\simeq \mathrm{R}\mathcal{H}om^{\mathrm{E}}\big(\mathbb{C}_{\widetilde{X}}^{\mathrm{E}}, \mathrm{DR}_{\widetilde{X}}^{\mathrm{E}}(\mathcal{N})\big)\\ &\simeq \mathrm{R}\mathcal{H}om^{\mathrm{E}}\big(\mathbb{C}_{\widetilde{X}}^{\mathrm{E}}, \Omega_{\widetilde{X}}^{\mathrm{E}} \otimes^{\mathrm{L}}_{\beta\pi^{-1}\mathcal{D}^{\mathcal{A}}_{\widetilde{X}}} \beta\pi^{-1}\mathcal{N}\big)\\ &\simeq \mathrm{R}\mathcal{H}om^{\mathrm{E}}(\mathbb{C}_{\widetilde{X}}^{\mathrm{E}}, \Omega_{\widetilde{X}}^{\mathrm{E}}) \otimes^{\mathrm{L}}_{\mathcal{D}^{\mathcal{A}}_{\widetilde{X}}} \mathcal{N},\end{aligned}$$

where the last isomorphism follows with [KS01, Theorem 5.6.1(ii)]. Therefore, it suffices to show that

$$\mathrm{R}\mathcal{H}om^{\mathrm{E}}(\mathbb{C}_{\widetilde{X}}^{\mathrm{E}}, \Omega_{\widetilde{X}}^{\mathrm{E}}) \simeq \varpi^{-1}\Omega_X \otimes_{\varpi^{-1}\mathcal{O}_X} \mathcal{A}_{\widetilde{X}}^{\mathrm{mod}}.$$

For this, we observe that

$$\begin{aligned}\mathrm{R}\mathcal{H}om^{\mathrm{E}}(\mathbb{C}_{\widetilde{X}}^{\mathrm{E}}, \mathcal{O}_{\widetilde{X}}^{\mathrm{E}}) &\simeq \mathrm{R}\mathcal{H}om^{\mathrm{E}}\big(\mathbb{C}_{\widetilde{X}}^{\mathrm{E}}, \mathrm{R}\mathcal{I}hom(\widetilde{\varpi}^{-1}\pi^{-1}\mathbb{C}_{X\setminus D}, \mathrm{E}\varpi^{!}\mathcal{O}_X^{\mathrm{E}})\big)\\ &\simeq \mathrm{R}\mathcal{H}om^{\mathrm{E}}\big(\widetilde{\varpi}^{-1}\pi^{-1}\mathbb{C}_{X\setminus D} \otimes \mathbb{C}_{\widetilde{X}}^{\mathrm{E}}, \mathrm{E}\varpi^{!}\mathcal{O}_X^{\mathrm{E}}\big)\\ &\simeq \mathrm{R}\mathcal{H}om^{\mathrm{E}}\big(\mathrm{E}\varpi^{-1}(\pi^{-1}\mathbb{C}_{X\setminus D} \otimes \mathbb{C}_X^{\mathrm{E}}), \mathrm{E}\varpi^{!}\mathcal{O}_X^{\mathrm{E}}\big)\\ &\simeq \mathrm{R}\mathcal{H}om^{\mathrm{E}}\big(\mathrm{E}\varpi^{-1}\mathcal{S}ol_X^{\mathrm{E}}(\mathcal{O}_X(*D)), \mathrm{E}\varpi^{!}\mathcal{O}_X^{\mathrm{E}}\big)\\ &\simeq \mathrm{R}\mathcal{H}om^{\mathrm{E}}\big(\mathcal{S}ol_{\widetilde{X}}^{\mathrm{E}}(\mathcal{O}_X(*D)^{\mathcal{A}}), \mathcal{O}_{\widetilde{X}}^{\mathrm{E}}\big)\\ &\simeq \mathcal{O}_X(*D)^{\mathcal{A}} \simeq \mathcal{A}_{\widetilde{X}}^{\mathrm{mod}}.\end{aligned}$$

Here, the first isomorphism follows from [DK16, (9.6.7)], the second-to-last line follows from the computations in [DK16, p. 192] and the last line follows from [DK16, (9.6.8)].

Consequently, we can conclude

$$\begin{aligned} \mathrm{R}\mathcal{H}om^{\mathrm{E}}\big(\mathbb{C}^{\mathrm{E}}_{\widetilde{X}}, \Omega^{\mathrm{E}}_{\widetilde{X}}\big) &\simeq \mathrm{R}\mathcal{H}om^{\mathrm{E}}\big(\mathbb{C}^{\mathrm{E}}_{\widetilde{X}}, \pi^{-1}\varpi^{-1}\Omega_X \otimes_{\pi^{-1}\varpi^{-1}\mathcal{O}_X} \mathcal{O}^{\mathrm{E}}_{\widetilde{X}}\big) \\ &\simeq \varpi^{-1}\Omega_X \otimes_{\varpi^{-1}\mathcal{O}_X} \mathrm{R}\mathcal{H}om^{\mathrm{E}}\big(\mathbb{C}^{\mathrm{E}}_{\widetilde{X}}, \mathcal{O}^{\mathrm{E}}_{\widetilde{X}}\big) \\ &\simeq \varpi^{-1}\Omega_X \otimes_{\varpi^{-1}\mathcal{O}_X} \mathcal{A}^{\mathrm{mod}}_{\widetilde{X}}, \end{aligned}$$

where the second isomorphism is due to [DK16, Lemma 4.10.3]. □

Proof of Proposition 3.1. Again we write $\widetilde{X}$, ϖ instead of $\widetilde{X}_D$, ϖ_D. For the left-hand side, one has

$$\begin{aligned} \mathsf{sh}_{\widetilde{X}}\big(\mathrm{E}\tilde{j}_{D*}\mathrm{E}j^{-1}\mathrm{DR}^{\mathrm{E}}_X(\mathcal{M})\big) &\simeq \mathsf{sh}_{\widetilde{X}}\big(\mathrm{E}\tilde{j}_{D*}\mathrm{E}j^{!}\mathrm{DR}^{\mathrm{E}}_X(\mathcal{M})\big) \\ &\simeq \mathsf{sh}_{\widetilde{X}}\big(\mathrm{E}j_*\mathrm{E}j^{!}\mathrm{E}\varpi^{!}\mathrm{DR}^{\mathrm{E}}_X(\mathcal{M})\big) \\ &\simeq \mathsf{sh}_{\widetilde{X}}\mathrm{R}\mathcal{I}hom\big(\pi^{-1}\mathbb{C}_{X\setminus D}, \mathrm{E}\varpi^{!}\mathrm{DR}^{\mathrm{E}}_X(\mathcal{M})\big) \\ &\simeq \mathsf{sh}_{\widetilde{X}}\mathrm{R}\mathcal{I}hom\big(\widetilde{\varpi}^{-1}\pi^{-1}\mathbb{C}_{X\setminus D}, \mathrm{E}\varpi^{!}\mathrm{DR}^{\mathrm{E}}_X(\mathcal{M})\big) \\ &\simeq \mathsf{sh}_{\widetilde{X}}\mathrm{E}\varpi^{!}\mathrm{R}\mathcal{I}hom\big(\pi^{-1}\mathbb{C}_{X\setminus D}, \mathrm{DR}^{\mathrm{E}}_X(\mathcal{M})\big) \\ &\simeq \mathsf{sh}_{\widetilde{X}}\mathrm{DR}^{\mathrm{E}}_{\widetilde{X}}(\mathcal{M}^{\mathcal{A}}). \end{aligned}$$

In the fourth isomorphism, we have used [DK19, Lemma 2.7.6] and the last isomorphism follows from [DK16, §9] (cf. also [IT20, p. 13]).

Since, by definition, we have $\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}}(\mathcal{M}) = \mathrm{DR}_{\widetilde{X}}(\mathcal{M}^{\mathcal{A}})$, we can now conclude with Lemma 3.2. □

One can deduce a similar result for the moderate de Rham functor on the real blow-up along the function defining a normal crossing divisor.

Corollary 3.3. *Let $f\colon X \to \mathbb{C}$ be a holomorphic function such that $D = f^{-1}(0) \subset X$ is a simple normal crossing divisor. Then there is an isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\widetilde{X}_f})$*

$$\mathsf{sh}_{\widetilde{X}_f}\big(\mathrm{E}\tilde{j}_{f*}\mathrm{E}j^{-1}\mathrm{DR}^{\mathrm{E}}_X(\mathcal{M})\big) \simeq \mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}_f}(\mathcal{M}).$$

Proof. There is a natural morphism $\varpi_{D,f}\colon \widetilde{X}_D \to \widetilde{X}_f$ and one has $\tilde{j}_f = \varpi_{D,f} \circ \tilde{j}_D$. Consequently,

$$\begin{aligned} \mathsf{sh}_{\widetilde{X}_f}\big(\mathrm{E}\tilde{j}_{f*}\mathrm{E}j^{-1}\mathrm{DR}^{\mathrm{E}}_X(\mathcal{M})\big) &\simeq \mathrm{R}\varpi_{D,f*}\mathsf{sh}_{\widetilde{X}_D}\big(\mathrm{E}\tilde{j}_{D*}\mathrm{E}j^{-1}\mathrm{DR}^{\mathrm{E}}_X(\mathcal{M})\big) \\ &\simeq \mathrm{R}\varpi_{D,f*}\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}_D}(\mathcal{M}) \simeq \mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}_f}(\mathcal{M}). \end{aligned}$$

Here, the first isomorphism follows from [DK21b, Lemma 3.9] since $\varpi_{D,f}$ is proper. The last isomorphism is proved in [Moc14, Proposition 4.7.4]. □

3.3. **Moderate de Rham complexes along a function.** In the previous section, we have studied the moderate de Rham complex on real blow-ups $\widetilde{X}_D$ and $\widetilde{X}_f$ in the case of a normal crossing divisor. To do this, we could directly apply the constructions performed in [DK16].

If X is a complex manifold and $f\colon X \to \mathbb{C}$ is a holomorphic function, the divisor $f^{-1}(0)$ does not need to have normal crossings. The blow-up space $\widetilde{X}_f$ can still be defined, but Corollary 3.3 does not apply to this case. In this subsection, we define a version of the enhanced de Rham functor on $\widetilde{X}_f$ in order to prove an analogous statement without the normal crossing assumption. This works along the same lines as the version for $\widetilde{X}_D$ in [DK16] (simply denoted by $\widetilde{X}$ in loc. cit.), but uses some interesting facts about resolutions of singularities.

Throughout this subsection, X will be a complex manifold and $f\colon X \to \mathbb{C}$ a holomorphic function. We write $X^* := X \setminus f^{-1}(0)$. We denote by $\widetilde{X}_f$ the real blow-up of X along f. Recall the notation for morphisms from (1). We set $\mathcal{D}^{\mathcal{A}}_{\widetilde{X}_f} := \mathcal{A}^{\mathrm{mod}}_{\widetilde{X}_f} \otimes_{\varpi^{-1}\mathcal{O}_X} \varpi^{-1}\mathcal{D}_X$. Let $\overline{X}$ denote the complex manifold conjugate to X.

The first step is to define a sheaf of tempered holomorphic functions on $\widetilde{X}_f$, by analogy with [DK16, §7.2].

Lemma-Definition 3.4. *We define the objects*

$$\mathcal{D}b^{\mathrm{t}}_{\widetilde{X}_f} := \varpi_f^! \mathrm{R}\mathcal{I}hom(\mathbb{C}_{X^*}, \mathcal{D}b^{\mathrm{t}}_X)$$

and

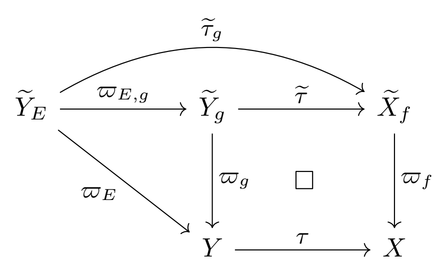
$$\mathcal{O}^{\mathrm{t}}_{\widetilde{X}_f} := \mathrm{R}\mathcal{H}om_{\varpi_f^{-1}\mathcal{D}_{\overline{X}}}(\varpi^{-1}\mathcal{O}_{\overline{X}}, \mathcal{D}b^{\mathrm{t}}_{\widetilde{X}_f}),$$

and the latter is an object in $\mathrm{D}^{\mathrm{b}}(\mathcal{D}^{\mathcal{A}}_{\widetilde{X}_f})$.

Proof. It is clear a priori that $\mathcal{D}b^{\mathrm{t}}_{\widetilde{X}_f}$ thus defined is an object over $\varpi^{-1}\mathcal{D}_X \otimes \varpi^{-1}\mathcal{D}_{\overline{X}}$, and hence that $\mathcal{O}^{\mathrm{t}}_{\widetilde{X}_f}$ is a module over $\varpi^{-1}\mathcal{D}_X$. It thus remains to show that it is an $\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}_f}$-module.

(a) Set $D := f^{-1}(0)$. Assume that we can find a modification $\tau\colon Y \to X$ (i.e., a proper morphism such that $E := \tau^{-1}(D)$ has simple normal crossings and induces an isomorphism $Y \setminus E \simeq X \setminus D$) and set $g := f \circ \tau$. Let us write $X^* := X \setminus D$ and $Y^* := Y \setminus E$. Let $\varpi_{E,g}\colon \widetilde{Y}_E \to \widetilde{Y}_g$ denote the natural proper morphism between the two real blow-ups on Y associated to $E = g^{-1}(0)$, denote by $\widetilde{\tau}\colon \widetilde{Y}_g \to \widetilde{X}_f$ the map induced by τ, and let $\widetilde{\tau}_g = \widetilde{\tau} \circ \varpi_{E,g}\colon \widetilde{Y}_E \to \widetilde{X}_f$ denote the composition. We have the following commutative diagram, where the square is Cartesian:

(4)



Since E is a normal crossing divisor, we know from [DK16, Notation 7.2.4 and Theorem 7.2.7] that

$$\mathcal{O}^{\mathrm{t}}_{\widetilde{Y}_E} \simeq \varpi_E^! \mathrm{R}\mathcal{I}hom(\mathbb{C}_{Y^*}, \mathcal{O}^{\mathrm{t}}_Y)$$

is an $\mathcal{A}^{\mathrm{mod}}_{\widetilde{Y}_E}$-module.

Moreover, we have isomorphisms

$$\begin{aligned}
\mathrm{R}\widetilde{\tau}_{g\,*}\mathcal{O}^{\mathrm{t}}_{\widetilde{Y}_E} &\simeq \mathrm{R}\widetilde{\tau}_*\varpi_{E,g_*}\mathrm{R}\mathcal{I}hom(\mathbb{C}_{Y^*},\varpi_E^!\mathcal{O}^{\mathrm{t}}_Y)\\
&\simeq \mathrm{R}\widetilde{\tau}_*\mathrm{R}\varpi_{E,g_*}\mathrm{R}\mathcal{I}hom(\mathbb{C}_{Y^*},\varpi_{E,g}^!\varpi_g^!\mathcal{O}^{\mathrm{t}}_Y)\\
&\simeq \mathrm{R}\widetilde{\tau}_*\mathrm{R}\mathcal{I}hom(\mathrm{R}\varpi_{E,g_*}\mathbb{C}_{Y^*},\varpi_g^!\mathcal{O}^{\mathrm{t}}_Y)\\
&\simeq \mathrm{R}\widetilde{\tau}_*\mathrm{R}\mathcal{I}hom(\mathbb{C}_{Y^*},\varpi_g^!\mathcal{O}^{\mathrm{t}}_Y)\\
&\simeq \mathrm{R}\mathcal{I}hom(\mathbb{C}_{X^*},\mathrm{R}\widetilde{\tau}_*\varpi_g^!\mathcal{O}^{\mathrm{t}}_Y)\\
&\simeq \mathrm{R}\mathcal{I}hom(\mathbb{C}_{X^*},\mathrm{R}\varpi_f^!\tau_*\mathcal{O}^{\mathrm{t}}_Y)\\
&\simeq \varpi_f^!\mathrm{R}\mathcal{I}hom(\mathbb{C}_{X^*},\mathrm{R}\tau_*\mathcal{O}^{\mathrm{t}}_Y)\\
&\simeq \varpi_f^!\mathrm{R}\mathcal{I}hom(\mathbb{C}_{X^*},\mathcal{O}^{\mathrm{t}}_X)\simeq \mathcal{O}^{\mathrm{t}}_{\widetilde{X}_f},
\end{aligned}$$

Here, we repeatedly use adjunction isomorphisms like [KS01, Proposition 5.3.8, Corollary 5.3.5] (together with the fact that all the morphisms in (4) are isomorphisms outside the given divisors). The sixth isomorphism uses the [KS01, Theorem 5.3.10], relying on the fact that the square in (4) is Cartesian. The last line follows from the tempered Grauert theorem [KS16, Theorem 3.1.5].

Consequently, $\mathcal{O}^{\mathrm{t}}_{\widetilde{X}_f}$ is a module (or, in general, a complex of modules) over $\widetilde{\tau}_{g\,*}\mathcal{A}^{\mathrm{mod}}_{\widetilde{Y}_E}\simeq \mathrm{R}\widetilde{\tau}_{g\,*}\mathcal{A}^{\mathrm{mod}}_{\widetilde{Y}_E}\simeq \mathcal{A}^{\mathrm{mod}}_{\widetilde{X}_f}$. This isomorphism follows from [Moc14, Theorem 4.1.5].

(b) Let us show that the action of $\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}_f}$ on $\mathcal{O}^{\mathrm{t}}_{\widetilde{X}_f}$ constructed in (a) is canonical and does not depend on the choice of the modification $\tau\colon Y\to X$.

First, we note that, similarly to the computation in (a), one obtains

$$\begin{aligned}
\widetilde{\tau}_g^!\mathcal{D}b^{\mathrm{t}}_{\widetilde{X}_f} &= \widetilde{\tau}_g^!\varpi_f^!\mathrm{R}\mathcal{I}hom(\mathbb{C}_{X^*},\mathcal{D}b^{\mathrm{t}}_X)\\
&\simeq \varpi_E^!\tau^!\mathrm{R}\mathcal{I}hom(\mathbb{C}_{X^*},\mathcal{D}b^{\mathrm{t}}_X)\\
&\simeq \varpi_E^!\mathrm{R}\mathcal{I}hom(\mathbb{C}_{Y^*},\mathcal{D}b^{\mathrm{t}}_Y)\\
&\simeq \mathcal{D}b^{\mathrm{t}}_{\widetilde{Y}_E}.
\end{aligned}$$

The second-to-last isomorphism is due to [KS16, Lemma 2.5.7], and the last isomorphism follows from [DK16, Theorem 7.2.7] (since E is a normal crossing divisor). Consequently, one has an isomorphism $\mathcal{D}b^{\mathrm{t}}_{\widetilde{Y}_E}\simeq \widetilde{\tau}_g^!\mathrm{R}\widetilde{\tau}_{g\,*}\mathcal{D}b^{\mathrm{t}}_{\widetilde{Y}_E}$.

Now, observe that

$$\begin{aligned}
\mathcal{O}^{\mathrm{t}}_{\widetilde{X}_f} &\simeq \mathrm{R}\widetilde{\tau}_{g\,*}\mathcal{O}^{\mathrm{t}}_{\widetilde{Y}_E}\simeq \mathrm{R}\widetilde{\tau}_{g\,*}\mathrm{R}\mathcal{H}om_{\varpi_E^{-1}\mathcal{D}_{\overline{Y}}}(\varpi_E^{-1}\mathcal{O}_{\overline{Y}},\mathcal{D}b^{\mathrm{t}}_{\widetilde{Y}_E})\\
&\simeq \mathrm{R}\widetilde{\tau}_{g\,*}\mathrm{R}\mathcal{H}om_{\varpi_E^{-1}\mathcal{D}_{\overline{Y}}}(\varpi_E^{-1}\mathcal{O}_{\overline{Y}},\widetilde{\tau}_g^!\mathrm{R}\widetilde{\tau}_{g\,*}\mathcal{D}b^{\mathrm{t}}_{\widetilde{Y}_E})\\
&\simeq \mathrm{R}\mathcal{H}om_{\widetilde{\tau}_{g\,*}\varpi_E^{-1}\mathcal{D}_{\overline{Y}}}(\mathrm{R}\widetilde{\tau}_{g\,*}\varpi_E^{-1}\mathcal{O}_{\overline{Y}},\mathrm{R}\widetilde{\tau}_{g\,*}\mathcal{D}b^{\mathrm{t}}_{\widetilde{Y}_E}),
\end{aligned}$$

showing that the $\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}_f}$-action on $\mathcal{O}^{\mathrm{t}}_{\widetilde{X}_f}$ is induced by the $\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}_f}$-action on $\mathrm{R}\widetilde{\tau}_{g\,*}\mathcal{D}b^{\mathrm{t}}_{\widetilde{Y}_E}$, which in turn is induced by the $\mathcal{A}^{\mathrm{mod}}_{\widetilde{Y}_E}$-action on $\mathcal{D}b^{\mathrm{t}}_{\widetilde{Y}_E}$. We remark that $\mathrm{R}\widetilde{\tau}_{g\,*}\mathcal{D}b^{\mathrm{t}}_{\widetilde{Y}_E}=\widetilde{\tau}_{g\,*}\mathcal{D}b^{\mathrm{t}}_{\widetilde{Y}_E}$ since tempered distributions form a quasi-injective object (cf. [Kas84, Theorem 3.18], [KS01, §7.2]).

In fact, the action of $\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}_f}\simeq \widetilde{\tau}_{g\,*}\mathcal{A}^{\mathrm{mod}}_{\widetilde{Y}_E}$ on $\widetilde{\tau}_{g\,*}\mathcal{D}b^{\mathrm{t}}_{\widetilde{Y}_E}$ does not depend on the choice of the projective morphism τ above: Let $U\subset\widetilde{X}_f$ be a subanalytic subset. A section of $\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}_f}$ on U is a holomorphic function, say φ, on $U \cap X^* = \pi^{-1}(U) \cap Y^*$ that has tempered growth along $U \cap \partial\widetilde{X}_f$ (or along $\tau^{-1}(U) \cap \partial\widetilde{Y}_E$, which is equivalent, since $\widetilde{\tau}_{g*}\mathcal{A}^{\mathrm{mod}}_{\widetilde{Y}_E} \simeq \mathcal{A}^{\mathrm{mod}}_{\widetilde{X}_f}$). On the other hand, we have $(\widetilde{\tau}_{g*}\mathcal{D}b^{\mathrm{t}}_{\widetilde{Y}_E})(U) = \mathcal{D}b^{\mathrm{t}}_{\widetilde{Y}_E}(\widetilde{\tau}_g^{-1}(U))$, so an element δ thereof is by definition a (tempered) distribution on $\widetilde{\tau}_g^{-1}(U) \cap Y^* = U \cap X^*$. The action of φ on δ is given by the natural multiplication of a distribution by a function, all defined outside the divisors. Hence, the induced action is independent of the choice of τ.

(c) In general, a modification similar to that in (a) exists by now well-known results on desingularization of analytic spaces (see, e.g., [AHV18]). However, E might not have smooth components, but we know that, locally around each point $x \in X$, we can find such a modification. Therefore, (a) shows that $\mathcal{O}^{\mathrm{t}}_{\widetilde{X}_f}$ has locally an action of $\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}_f}$. Since the local actions are canonical (and hence compatible) by (b), they give a global action of $\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}_f}$ on $\mathcal{O}^{\mathrm{t}}_{\widetilde{X}_f}$.

□

We can now define (using notation similar to the one in [DK16, §9.2])

$$\mathcal{O}^{\mathrm{E}}_{\widetilde{X}_f} := \widetilde{i}^! \mathrm{R}\mathcal{H}om_{\mathcal{D}_{\mathbb{P}}}(\mathcal{E}^{\tau}_{\mathbb{C}|\mathbb{P}}, \mathcal{O}^{\mathrm{t}}_{\widetilde{X}_f \times \mathbb{P}})[2],$$
$$\Omega^{\mathrm{E}}_{\widetilde{X}_f} := \pi^{-1}_{\widetilde{X}_f}\varpi_f^{-1}\Omega_X \otimes^{\mathrm{L}}_{\pi^{-1}_{\widetilde{X}_f}\varpi_f^{-1}\mathcal{O}_X} \mathcal{O}^{\mathrm{E}}_{\widetilde{X}_f}$$

and one sets

$$\mathrm{DR}^{\mathrm{E}}_{\widetilde{X}_f}(\mathcal{L}) := \Omega^{\mathrm{E}}_{\widetilde{X}_f} \otimes^{\mathrm{L}}_{\mathcal{D}^{\mathcal{A}}_{\widetilde{X}_f}} \mathcal{L} \qquad \text{for } \mathcal{L} \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}^{\mathcal{A}}_{\widetilde{X}_f}).$$

As in loc. cit. we then get

$$\mathrm{DR}^{\mathrm{E}}_{\widetilde{X}_f}(\mathcal{M}^{\mathcal{A}_f}) \simeq \mathrm{E}\varpi_f^! \mathrm{DR}^{\mathrm{E}}_X(\mathcal{M}).$$

for any meromorphic connection $\mathcal{M}$ (setting $\mathcal{M}^{\mathcal{A}_f} := \mathcal{A}^{\mathrm{mod}}_{\widetilde{X}_f} \otimes_{\varpi_f^{-1}\mathcal{O}_X} \varpi_f^{-1}\mathcal{M}$).

Then we can reproduce the proof of Lemma 3.2 and Proposition 3.1 along the exact same lines and obtain the following.

Proposition 3.5. *Let X be a complex manifold and $f\colon X \to \mathbb{C}$ a holomorphic function. Then there is an isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\widetilde{X}_f})$*

$$\mathsf{sh}_{\widetilde{X}_f}\big(\mathrm{E}\widetilde{j}_{f*}\mathrm{E}j^{-1}\mathrm{DR}^{\mathrm{E}}_X(\mathcal{M})\big) \simeq \mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}_f}(\mathcal{M}).$$

Proof. The main steps of this proof are the isomorphisms

$$\mathcal{O}^{\mathrm{E}}_{\widetilde{X}_f} \simeq \mathrm{R}\mathcal{I}hom(\widetilde{\varpi}^{-1}\pi^{-1}\mathbb{C}_{X^*}, \mathrm{E}\varpi^! \mathcal{O}^{\mathrm{E}}_X)$$

and

$$\mathrm{E}\varpi^{-1}\mathcal{S}ol^{\mathrm{E}}_X\big(\mathcal{O}_X(*D)\big) \simeq \mathcal{S}ol^{\mathrm{E}}_{\widetilde{X}_f}\big(\mathcal{O}_X(*D)^{\mathcal{A}_f}\big),$$

which we both derive from Lemma-Definition 3.4 as in the case of $\widetilde{X}_D$ in [DK16], and the isomorphism

$$\alpha_{\widetilde{X}_f}\mathcal{O}^{\mathrm{t}}_{\widetilde{X}_f} \simeq \mathcal{A}^{\mathrm{mod}}_{\widetilde{X}_f},$$

and this follows again from a local consideration as in part (a) of the above proof since $\alpha_{\widetilde{X}_f}$ is compatible with direct images (see [KS01, Proposition 4.3.6]) and we know such a statement for the blow-up $\widetilde{Y}_E$ along a normal crossing divisor (see [DK16, Proposition 7.2.10]). □

3.4. **Rapid decay de Rham complexes: a motivation in dimension one.** So far we have studied moderate de Rham complexes. The aim is now to find a functorial way to extract the rapid decay de Rham complex from the enhanced de Rham complex. Rapid decay functions do not seem to have been studied in the context of enhances ind-sheaves.

Studying the one-dimensional case again, it is quite straightforward to find an expression similar to the one for the moderate de Rham complex. Let X be a complex manifold. Recall the notation from Section 3.1.

Lemma 3.6. *For an object $K \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X)$, one has a distinguished triangle in $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\widetilde{X}})$*

$$\mathrm{E}\tilde{j}_{!!}\mathrm{E}j^{-1}K \longrightarrow \mathrm{E}\tilde{j}_{*}\mathrm{E}j^{-1}K \longrightarrow \mathrm{E}i_{!!}\mathrm{E}i^{-1}\mathrm{E}\tilde{j}_{*}\mathrm{E}j^{-1}K \xrightarrow{+1}$$

Proof. By [DK19, Lemma 2.7.6], we know that there are isomorphisms

$$\mathrm{E}\tilde{j}_{!!}\mathrm{E}\tilde{j}^{-1}H \simeq \pi^{-1}\mathbb{C}_{X\setminus D} \otimes H$$
$$\mathrm{E}i_{!!}\mathrm{E}i^{-1}H \simeq \pi^{-1}\mathbb{C}_{\partial\widetilde{X}} \otimes H$$

for any $H \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\widetilde{X}})$.

From the natural short exact sequence (cf. [KS90, Proposition 2.3.6(v)])

$$0 \longrightarrow \mathbb{C}_{X\setminus D} \longrightarrow \mathbb{C}_{\widetilde{X}} \longrightarrow \mathbb{C}_{\partial\widetilde{X}} \longrightarrow 0$$

in $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\widetilde{X}})$, we therefore obtain (applying the functor $\pi^{-1}(\bullet) \otimes \mathrm{E}\tilde{j}_*\mathrm{E}j^{-1}K$) a distinguished triangle

$$\mathrm{E}\tilde{j}_{!!}\mathrm{E}\tilde{j}^{-1}\mathrm{E}\tilde{j}_{*}\mathrm{E}j^{-1}K \longrightarrow \mathrm{E}\tilde{j}_{*}\mathrm{E}j^{-1}K \longrightarrow \mathrm{E}i_{!!}\mathrm{E}i^{-1}\mathrm{E}\tilde{j}_{*}\mathrm{E}j^{-1}K \xrightarrow{+1}$$

and this is the desired triangle, noting that $\mathrm{E}j^{-1}\mathrm{E}j_* \simeq \mathrm{E}j^!\mathrm{E}j_* \simeq \mathrm{id}$. □

Proposition 3.7. *Let $K \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X)$, then there is a short exact sequence*

$$0 \longrightarrow i^{-1}\mathsf{sh}_{\widetilde{X}}(\mathrm{E}\tilde{j}_{!!}\mathrm{E}j^{-1}K) \longrightarrow \Psi_a^{\leq 0}(K) \longrightarrow \Psi_a^0(K) \longrightarrow 0.$$

Proof. We apply the functor $i^{-1}\mathsf{sh}_{\widetilde{X}}(\bullet)$ to the distinguished triangle from Lemma 3.6. Then the middle object becomes $\Psi_a^{\leq 0}(K)$. Moreover, the third object becomes $\Psi_a^0(K)$, noting that $\mathsf{sh}_{\widetilde{X}} \circ \mathrm{E}i_{!!} \simeq i_! \circ \mathsf{sh}_{\partial\widetilde{X}}$ since i is proper (see [DK21b, Lemma 3.9(ii)]) and $i^{-1}i_! \simeq \mathrm{id}$.

The distinguished triangle thus obtained is indeed a short exact sequence since we already know that the second and third objects are concentrated in degree zero and the morphism between them is an epimorphism (see [DK20, Lemma 4.3]), which implies that also the first object must be concentrated in degree 0. □

Now let $\mathcal{M}$ a meromorphic connection with a pole at 0, and let $K = \mathrm{DR}_X^{\mathrm{E}}(\mathcal{M})$ and $\mathcal{L}_{\mathcal{M}} := i^{-1}\tilde{j}_*j^{-1}\mathrm{DR}_X(\mathcal{M})[-1]$. Then the objects in the short exact sequence of Proposition 3.7 are related to the Stokes filtration by

$$\Psi_a^{\leq 0}(K) \simeq F_{\leq 0}\mathcal{L}_{\mathcal{M}},$$
$$\Psi_a^{0}(K) \simeq \mathrm{Gr}_0^F\mathcal{L}_{\mathcal{M}} = F_{\leq 0}\mathcal{L}_{\mathcal{M}}/F_{<0}\mathcal{L}_{\mathcal{M}}.$$

Therefore, this implies that there is an isomorphism

$$F_{<0}\mathcal{L}_{\mathcal{M}} \simeq i^{-1}\mathsf{sh}_{\widetilde{X}}(\mathrm{E}\tilde{j}_{!!}\mathrm{E}j^{-1}K).$$

Since the "< 0" part of the Stokes filtration is nothing but the rapid decay de Rham complex, restricted to $\partial\widetilde{X}$, this implies that it is reasonable to expect an isomorphism

$$\mathrm{DR}^{\mathrm{rd}}_{\widetilde{X}}(\mathcal{M}) \simeq \mathsf{sh}_{\widetilde{X}}\big(\mathrm{E}\tilde{\jmath}_{!!}\mathrm{E}j^{-1}\mathrm{DR}^{\mathrm{E}}_{X}(\mathcal{M})\big), \tag{5}$$

a formula similar to the ones proved for moderate de Rham complexes above.

Remark 3.8. Note that the functor $\mathsf{sh}_{\widetilde{X}} \circ \mathrm{E}\tilde{\jmath}_{!!} \not\simeq \tilde{\jmath}_{!} \circ \mathsf{sh}_{X^*_\infty}$, in particular this functor is not an "extension by zero" as known from the theory of classical sheaves.

A proof of this formula (5) in the case that f defines a normal crossing divisor (in any dimension) will be given in Section 6 (see Corollary 6.11), using deep results on duality of certain sheaves of functions.

4. Moderate and rapid decay objects associated to enhanced ind-sheaves

In Section 3, we have studied functorial ways to extract the moderate growth and rapid decay de Rham complexes from the enhanced de Rham complex. Inspired by these considerations, we define here moderate growth and rapid decay objects (more precisely, sheaves on the real blow-up space) to any enhanced ind-sheaf on X and establish some of their basic properties.

Throughout this section, let X be a complex manifold and let $f\colon X \to \mathbb{C}$ be holomorphic function. Since we will only work on the blow-up $\widetilde{X}_f$ from here on, we will simplify the notation for the morphisms from diagram (1) by suppressing the index f as follows:

$$\partial\widetilde{X}_f \xhookrightarrow{\ i\ } \widetilde{X}_f \xhookleftarrow{\ \tilde{\jmath}\ } X^*_\infty \xhookrightarrow{\ j\ } X, \qquad \varpi\colon \widetilde{X}_f \to X$$

We will also work over an arbitrary field k here.

Definition 4.1. *For an enhanced ind-sheaf $K \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}k_X)$, we define the objects of* $\mathrm{D}^{\mathrm{b}}(k_{\widetilde{X}})$

$$K^{\mathrm{mod}\, f} := \mathsf{sh}_{\widetilde{X}_f}\big(\mathrm{E}\tilde{\jmath}_{*}\mathrm{E}j^{-1}K\big), \tag{6}$$

$$K^{\mathrm{rd}\, f} := \mathsf{sh}_{\widetilde{X}_f}\big(\mathrm{E}\tilde{\jmath}_{!!}\mathrm{E}j^{-1}K\big). \tag{7}$$

Remark 4.2. Let us note here that we could also perform the same construction on the real blow-up $\widetilde{X}_D$ along a normal crossing divisor and in this way define objects $K^{\mathrm{mod}\, D}$ and $K^{\mathrm{rd}\, D}$. We will, however, focus on the blow-up along a function from here on, although the constructions and results we prove in the present and the following section work analogously on $\widetilde{X}_D$. We will use this fact in the proofs of Proposition 6.5 and Proposition 6.12 below.

The first easy observation is the following.

Lemma 4.3. *If we write $X^* := X \setminus f^{-1}(0)$, there are isomorphisms for any $K \in$* $\mathrm{E}^{\mathrm{b}}(\mathrm{I}k_X)$

$$K^{\mathrm{mod}\, f} \simeq (\pi^{-1}k_{X^*} \otimes K)^{\mathrm{mod}\, f} \simeq \mathrm{R}\mathcal{I}hom(\pi^{-1}k_{X^*}, K)^{\mathrm{mod}\, f},$$

$$K^{\mathrm{rd}\, f} \simeq (\pi^{-1}k_{X^*} \otimes K)^{\mathrm{rd}\, f} \simeq \mathrm{R}\mathcal{I}hom(\pi^{-1}k_{X^*}, K)^{\mathrm{rd}\, f}.$$

Proof. It follows from [DK19, Lemma 2.7.6] that we have

$$\pi^{-1}k_{X^*} \otimes K \simeq \mathrm{E}j_{!!}\mathrm{E}j^{-1}K,$$
$$\mathrm{R}\mathcal{I}hom(k_{X^*}, K) \simeq \mathrm{E}j_*\mathrm{E}j^{-1}K.$$

(For the second isomorphism, note that j is an open embedding and hence $j^! = j^{-1}$.) Then, the statement of the lemma is clear from the definition because $\mathrm{E}j^{-1}\mathrm{E}j_* = \mathrm{id} = \mathrm{E}j^{-1}\mathrm{E}j_{!!}$. □

Remark 4.4. In the context of enhanced de Rham functors (in particular, $k = \mathbb{C}$ here), the previous lemma tells us that the functors from Definition 4.1 are not sensitive to localization of the D-module: If $D = f^{-1}(0)$ and $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$, then by [DK16, Theorem 9.1.2], we know that

$$\mathrm{DR}^{\mathrm{E}}_X\big(\mathcal{M}(*D)\big) \simeq \mathrm{R}\mathcal{I}hom\big(\pi^{-1}k_{X^*}, \mathrm{DR}^{\mathrm{E}}_X(\mathcal{M})\big),$$
$$\mathrm{DR}^{\mathrm{E}}_X\big(\mathcal{M}(!D)\big) \simeq \pi^{-1}k_{X^*} \otimes \mathrm{DR}^{\mathrm{E}}_X(\mathcal{M})$$

and hence the lemma means nothing but

$$\big(\mathrm{DR}^{\mathrm{E}}_X(\mathcal{M})\big)^{\mathrm{mod}\, f} \simeq \big(\mathrm{DR}^{\mathrm{E}}_X(\mathcal{M}(*D))\big)^{\mathrm{mod}\, f} \simeq \big(\mathrm{DR}^{\mathrm{E}}_X(\mathcal{M}(!D))\big)^{\mathrm{mod}\, f},$$
$$\big(\mathrm{DR}^{\mathrm{E}}_X(\mathcal{M})\big)^{\mathrm{rd}\, f} \simeq \big(\mathrm{DR}^{\mathrm{E}}_X(\mathcal{M}(*D))\big)^{\mathrm{rd}\, f} \simeq \big(\mathrm{DR}^{\mathrm{E}}_X(\mathcal{M}(!D))\big)^{\mathrm{rd}\, f}.$$

By restricting the objects from Definition 4.1 to the boundary of the real blow-up, we can define objects that will be called *enhanced nearby cycles*; they are higher-dimensional analogues of the constructions in [DK20] as well as analogues in the world of enhanced ind-sheaves of the constructions performed for D-modules in [Sab21].

Definition 4.5. *Let $K \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}k_X)$, then we set*

$$\psi_f^{\leq 0}K := i^{-1}K^{\mathrm{mod}\, f} = i^{-1}\mathsf{sh}_{\widetilde{X}_f}\big(\mathrm{E}\tilde{j}_*\mathrm{E}j^{-1}K\big)[-1],$$
$$\psi_f^{<0}K := i^{-1}K^{\mathrm{rd}\, f} = i^{-1}\mathsf{sh}_{\widetilde{X}_f}\big(\mathrm{E}\tilde{j}_{!!}\mathrm{E}j^{-1}K\big)[-1],$$
$$\psi_f^{>0}K := i^{!}K^{\mathrm{mod}\, f} = i^{!}\mathsf{sh}_{\widetilde{X}_f}\big(\mathrm{E}\tilde{j}_*\mathrm{E}j^{-1}K\big),$$
$$\psi_f^{\geq 0}K := i^{!}K^{\mathrm{rd}\, f} = i^{!}\mathsf{sh}_{\widetilde{X}_f}\big(\mathrm{E}\tilde{j}_{!!}\mathrm{E}j^{-1}K\big).$$

Moreover, set

$$\psi_f^{*}K := i^{-1}\mathrm{R}\tilde{j}_*j^{-1}\mathsf{sh}_X K[-1].$$

All of them are objects in $\mathrm{D}^{\mathrm{b}}(k_{\partial\widetilde{X}_f})$.

Remark 4.6. One can express these objects differently in the case where K is the enhanced de Rham or solution object of a meromorphic connection (and in particular $k = \mathbb{C}$).

For example, let $\mathcal{M}$ be a meromorphic connection with poles along the normal crossing divisor $D = f^{-1}(0)$. Then

$$\begin{aligned}\psi_f^{\leq 0}\mathrm{DR}^{\mathrm{E}}_X(\mathcal{M}) &\cong i^{-1}\mathsf{sh}_{\widetilde{X}_f}\big(\mathrm{E}\tilde{j}_*\mathrm{E}j^{-1}\mathrm{DR}^{\mathrm{E}}_X(\mathcal{M})\big)[-1]\\ &\cong i^{-1}\mathsf{sh}_{\widetilde{X}_f}\big(\mathrm{E}\varpi^!\mathrm{E}j_*\mathrm{E}j^{-1}\mathrm{DR}^{\mathrm{E}}_X(\mathcal{M})\big)[-1]\\ &\cong i^{-1}\mathsf{sh}_{\widetilde{X}_f}\big(\mathrm{E}\varpi^!\mathrm{DR}^{\mathrm{E}}_X(\mathcal{M})\big)[-1].\end{aligned}$$

Similarly, one can show $\psi_f^{>0}\mathrm{DR}^{\mathrm{E}}_X(\mathcal{M}) \cong i^!\mathsf{sh}_{\widetilde{X}_f}\big(\mathrm{E}\varpi^!\mathrm{DR}^{\mathrm{E}}_X(\mathcal{M})\big)$, which is the analogue of the complex ${}^p\psi_f^{>\mathrm{mod}}\mathcal{M}$ in [Sab21].

Along the same lines, we get $\psi_f^{<0}\mathcal{S}ol_X^{\mathrm{E}}(\mathcal{M}) \cong i^{-1}\mathsf{sh}_{\widetilde{X}_f}\big(\mathrm{E}\varpi^{-1}\mathcal{S}ol_X^{\mathrm{E}}(\mathcal{M})\big)[-1]$ and $\psi_f^{\geq 0}\mathcal{S}ol_X^{\mathrm{E}}(\mathcal{M}) \cong i^!\mathsf{sh}_{\widetilde{X}_f}\big(\mathrm{E}\varpi^{-1}\mathcal{S}ol_X^{\mathrm{E}}(\mathcal{M})\big)$.

Proposition 4.7. *We have the following natural distinguished triangles in the category* $\mathrm{D}^{\mathrm{b}}(k_{\partial\widetilde{X}_f})$*:*

$$\psi_f^{\leq 0}K \xrightarrow{\ v\ } \psi_f^{*}K \longrightarrow \psi_f^{>0}K \xrightarrow{+1}$$

$$\psi_f^{<0}K \longrightarrow \psi_f^{*}K \xrightarrow{\ c\ } \psi_f^{\geq 0}K \xrightarrow{+1}$$

Proof. We start with the natural distinguished triangle

$$\mathsf{sh}_{\widetilde{X}_f}(\mathrm{E}\tilde{j}_*\mathrm{E}j^{-1}K) \to R\tilde{j}_*\tilde{j}^{-1}\mathsf{sh}_{\widetilde{X}_f}(\mathrm{E}\tilde{j}_*\mathrm{E}j^{-1}K) \to i_!i^!\mathsf{sh}_{\widetilde{X}_f}(\mathrm{E}\tilde{j}_*\mathrm{E}j^{-1}K)[1] \xrightarrow{+1} \tag{8}$$

(cf., e.g., [KS90, Proposition 2.4.6]). Applying $i^{-1}[-1]$ and noting that we have natural isomorphisms (recall the compatibility of the sheafification functor with pullbacks along open embeddings from [DK21b, Lemma 3.9])

$$\begin{aligned} i^{-1}R\tilde{j}_*\tilde{j}^{-1}\mathsf{sh}_{\widetilde{X}_f}(\mathrm{E}\tilde{j}_*\mathrm{E}j^{-1}K) &\simeq i^{-1}R\tilde{j}_*\mathsf{sh}_{X^*_\infty}(\mathrm{E}\tilde{j}^{-1}\mathrm{E}\tilde{j}_*\mathrm{E}j^{-1}K) \\ &\simeq i^{-1}R\tilde{j}_*\mathsf{sh}_{X^*_\infty}(\mathrm{E}j^{-1}K) \\ &\simeq i^{-1}R\tilde{j}_*j^{-1}\mathsf{sh}_X(K), \end{aligned}$$

we obtain the first distinguished triangle.

The second triangle is found analogously, replacing $\mathrm{E}j_*$ by $\mathrm{E}j_{!!}$ in (8). □

Proposition 4.8. *There is a natural distinguished triangle in* $\mathrm{D}^{\mathrm{b}}(k_{\partial\widetilde{X}_f})$

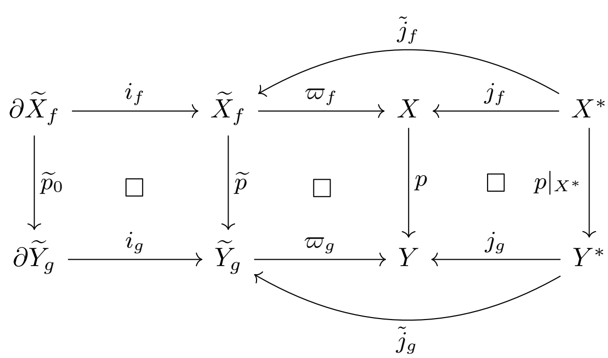
$$\psi_f^{<0}K \longrightarrow \psi_f^{\leq 0}K \longrightarrow \psi_f^{0}K \xrightarrow{+1}$$

where $\psi_f^0 K := \mathsf{sh}_{\partial\widetilde{X}_f}(\mathrm{E}i^{-1}\mathrm{E}\tilde{j}_*\mathrm{E}j^{-1}K)$.

Proof. This follows directly from Lemma 3.6. □

Let $p\colon X \to Y$ be a proper holomorphic map, $g\colon Y \to \mathbb{C}$ any holomorphic map, and set $f := g \circ p$. Then, p lifts to a morphism between the real blow-up spaces $\widetilde{p}\colon \widetilde{X}_f = X \times_Y \widetilde{Y}_g \to \widetilde{Y}_g$, and we obtain a natural commutative diagram where all the squares are Cartesian:

(9)



We have the following statement, which is the analogue of [Sab21, Proposition 6.6].

Proposition 4.9. *Let X, Y, f, g, and p be as above, and let $K \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}k_X)$. Then there is an isomorphism of distinguished triangles in $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_{\partial\widetilde{Y}_g})$:*

$$\begin{array}{ccccccc} \mathrm{R}\widetilde{p}_{0*}\psi_f^{\leq 0}K & \longrightarrow & \mathrm{R}\widetilde{p}_{0*}\psi_f^{*}K & \longrightarrow & \mathrm{R}\widetilde{p}_{0*}\psi_f^{>0}K & \xrightarrow{+1} \\ \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & \\ \psi_g^{\leq 0}\mathrm{E}p_*K & \longrightarrow & \psi_g^{*}\mathrm{E}p_*K & \longrightarrow & \psi_g^{>0}\mathrm{E}p_*K & \xrightarrow{+1} \end{array}$$

Similarly, there is an isomorphism

$$\begin{array}{ccccccc} \mathrm{R}\widetilde{p}_{0*}\psi_f^{<0}K & \longrightarrow & \mathrm{R}\widetilde{p}_{0*}\psi_f^{*}K & \longrightarrow & \mathrm{R}\widetilde{p}_{0*}\psi_f^{\geq 0}K & \xrightarrow{+1} \\ \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & \\ \psi_g^{<0}\mathrm{E}p_*K & \longrightarrow & \psi_g^{*}\mathrm{E}p_*K & \longrightarrow & \psi_g^{\geq 0}\mathrm{E}p_*K & \xrightarrow{+1} \end{array}$$

Proof. Since all the isomorphisms work along the same lines, let us only prove the first one (on the left of the first isomorphism of triangles).

$$\begin{aligned} \mathrm{R}\widetilde{p}_{0*}\psi_f^{\leq 0}K &\simeq \mathrm{R}\widetilde{p}_{0!}i_f^{-1}\mathsf{sh}_{\widetilde{X}_f}\big(\mathrm{E}\tilde{j}_{f*}\mathrm{E}j_f^{-1}K\big)[-1] \\ &\simeq i_g^{-1}\mathrm{R}\widetilde{p}_!\mathsf{sh}_{\widetilde{X}_f}\big(\mathrm{E}\tilde{j}_{f*}\mathrm{E}j_f^{-1}K\big)[-1] \\ &\simeq i_g^{-1}\mathsf{sh}_{\widetilde{X}_f}\big(\mathrm{E}\widetilde{p}_{!!}\mathrm{E}\tilde{j}_{f*}\mathrm{E}j_f^{-1}K\big)[-1] \\ &\simeq i_g^{-1}\mathsf{sh}_{\widetilde{X}_f}\big(\mathrm{E}\tilde{j}_{g*}\mathrm{E}(p|_{X^*})_{!!}\mathrm{E}j_f^{-1}K\big)[-1] \\ &\simeq i_g^{-1}\mathsf{sh}_{\widetilde{X}_f}\big(\mathrm{E}\tilde{j}_{g*}\mathrm{E}j_g^{-1}\mathrm{E}p_{!!}K\big)[-1] \simeq \psi_g^{\leq 0}\mathrm{E}p_*K. \end{aligned}$$

Here, we repeatedly used the facts that the squares in (9) are Cartesian (see [DK16, Proposition 4.5.11] for a base change formula for operations on enhanced ind-sheaves) and that p is proper and hence $\mathrm{E}p_* = \mathrm{E}p_{!!}$ (and similarly for $\widetilde{p}$).

The isomorphisms of objects thus obtained induce isomorphisms of distinguished triangles since the morphisms in the triangles of Proposition 4.7 are all canonical and the isomorphism constructed above is natural. □

5. Constructibility and duality

5.1. **Constructibility of moderate and rapid decay objects.** The following lemma is clear by construction.

Lemma 5.1. *For $K \in \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}k_X)$ (in particular for $K = \mathrm{DR}_X^{\mathrm{E}}(\mathcal{M})$ for a holonomic $\mathcal{D}_X$-module in the case $k = \mathbb{C}$), the objects defined in Definition 4.5 are $\mathbb{R}$-constructible.*

Proof. It suffices to note that the functors $\mathrm{E}\tilde{j}_{f*}$, $\mathrm{E}\tilde{j}_{f!!}$, $\mathrm{E}j^{-1}$ and $\mathsf{sh}_{\widetilde{X}_f}$ involved in the construction of $K^{\mathrm{mod}\, f}$ and $K^{\mathrm{rd}\, f}$ all preserve $\mathbb{R}$-constructibility (see [DK19, Proposition 3.3.3] and [KS16, Theorem 6.6.4]). □

5.2. **A short review on perfect pairings.** Let k be a field and let M be a locally compact topological space. We recall the following notion of a perfect pairing in the derived category $\mathrm{D}^{\mathrm{b}}(k_M)$. A good reference is [KS90], as well as [FSY21, Appendix C].

Let $F, G \in \mathrm{D}^{\mathrm{b}}(k_M)$, and let ω_M denote the Verdier dualizing complex. If $a_M\colon M \to \{\mathrm{pt}\}$ is the canonical map to the one-point space, then we have $\omega_M = a_M^! k$. The Verdier dual of $F \in \mathrm{D}^{\mathrm{b}}(k_M)$ is the object $\mathrm{D}_M F := \mathrm{R}\mathcal{H}om(F, \omega_M)$.

Definition 5.2. *Recall that a* pairing $\eta\colon F \otimes G \to \omega_M$ *is equivalent to the datum of a morphism* $F \to \mathrm{D}_M G$ *in* $\mathrm{D}^{\mathrm{b}}(k_M)$ *(or, equivalently, a morphism* $G \to \mathrm{D}_M F$*).*

We say that a pairing $\eta\colon F \otimes G \to \omega_M$ *is* perfect *if the associated morphism* $F \to \mathrm{D}_M G$ *(or* $G \to \mathrm{D}_M F$*) is an isomorphism.*

Now, suppose $\eta\colon F \otimes G \to \omega_M$ is such a perfect pairing with an isomorphism $G \simeq D_M F$. Taking derived global sections, we find

$$\begin{aligned} \mathrm{R}\Gamma(M; \mathrm{D}_M F) &\simeq \mathrm{RHom}(F, \omega_M) \\ &\simeq \mathrm{RHom}(\mathrm{R}\Gamma_c(M; F), k) \\ &=: \mathrm{R}\Gamma_c(M; F)^\vee. \end{aligned}$$

In particular, the hypercohomology of this object satisfies, for all $\ell \in \mathbb{Z}$,

$$\begin{aligned} \mathbb{H}^\ell(M; \mathrm{D}_M F) \simeq \mathrm{H}^0 \mathrm{R}\Gamma(M; (\mathrm{D}_M F)[\ell]) &\simeq \mathrm{H}^0\, \mathrm{RHom}(\mathrm{R}\Gamma_c(M; F), k)[\ell] \\ &\simeq \mathrm{H}^0\, \mathrm{RHom}(\mathrm{R}\Gamma_c(X; F)[-\ell], k) \\ &\simeq \mathbb{H}_c^{-\ell}(M; F)^\vee. \end{aligned}$$

By [KS90, (2.6.23)], there is a natural morphism

$$\mathrm{R}\Gamma(M; \mathrm{D}_M F) \otimes \mathrm{R}\Gamma_c(M; F) \to \mathrm{R}\Gamma_c(M; F \otimes \mathrm{D}_M F).$$

Taking hypercohomology and composing with the morphism induced by η on global sections, we obtain perfect pairings for all $\ell \in \mathbb{Z}$:

$$\mathbb{H}^\ell(M; \mathrm{D}_M F) \otimes \mathbb{H}_c^{-\ell}(M; F) \longrightarrow \mathbb{H}_c^0(M; \omega_M) \simeq k. \tag{10}$$

Proposition 5.3 ([FSY21, Corollary C.6])**.** *Let* $F, G \in \mathrm{D}^{\mathrm{b}}(k_M)$*. If the pairing*

$$\eta : F \otimes G \to \omega_M$$

is perfect, then so are the pairings

$$\eta^\ell : \mathbb{H}^\ell(M; F) \otimes \mathbb{H}_c^{-\ell}(M; G) \to k$$

for all $\ell \in \mathbb{Z}$*.*

We will refer to the pairings (10) as the global duality pairings of F (on hypercohomology). Similarly, we will refer to $\eta\colon F \otimes \mathrm{D}_M F \to \omega_M$ as the local duality pairing of F.

5.3. **Local duality statements for enhanced nearby cycles.** With our definition of the moderate growth and rapid decay objects, it is easy to see that for $\mathbb{R}$-constructible enhanced ind-sheaves the associated moderate and rapid decay objects are related by duality.

Let X be a complex manifold and $f\colon X \to \mathbb{C}$ a holomorphic function.

Proposition 5.4 (Local duality pairing on $\widetilde{X}_f$)**.** *Let* $K \in \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}k_X)$*. Then, there is a canonical isomorphism*

$$\mathrm{D}_{\widetilde{X}_f} K^{\mathrm{mod}\, f} \simeq (\mathrm{D}^{\mathrm{E}}_X K)^{\mathrm{rd}\, f}.$$

That is, there exists a perfect pairing

$$K^{\mathrm{mod}\, f} \otimes (\mathrm{D}^{\mathrm{E}}_X K)^{\mathrm{rd}\, f} \longrightarrow \omega_{\widetilde{X}_f},$$

where $\omega_{\widetilde{X}_f}$ *is the Verdier dualizing complex in* $\mathrm{D^b}(k_{\widetilde{X}_f})$*.*

Proof. There is the chain of isomorphisms

$$\begin{aligned}\mathrm{D}_{\widetilde{X}_f}\mathsf{sh}_{\widetilde{X}_f}\big(\mathrm{E}\tilde{j}_{f*}\mathrm{E}j^{-1}K\big) &\simeq \mathsf{sh}_{\widetilde{X}_f}\big(\mathrm{D}^{\mathrm{E}}_{\widetilde{X}_f}\mathrm{E}\tilde{j}_{f*}\mathrm{E}\tilde{j}^!_f\mathrm{E}\varpi^!K\big)\\ &\simeq \mathsf{sh}_{\widetilde{X}_f}\big(\mathrm{D}^{\mathrm{E}}_{\widetilde{X}_f}\mathrm{R}\mathcal{I}hom(\pi^{-1}k_{X^*},\mathrm{E}\varpi^!K)\big)\\ &\simeq \mathsf{sh}_{\widetilde{X}_f}\big(\pi^{-1}k_{X^*}\otimes\mathrm{D}^{\mathrm{E}}_{\widetilde{X}_f}\mathrm{E}\varpi^!\mathrm{DR}^{\mathrm{E}}_X(\mathcal{M})\big)\\ &\simeq \mathsf{sh}_{\widetilde{X}_f}\big(\mathrm{E}j^f_{!!}\mathrm{E}j^{f\,-1}\mathrm{E}\varpi^{-1}\mathrm{D}^{\mathrm{E}}_XK\big),\end{aligned}$$

where the first isomorphism follows from the commutation of sheafification with duality (see [DK21b, Lemma 3.13]), the third isomorphism follows from [DK16, Lemma 4.3.2, Proposition 4.9.13], and in the last isomorphism we have used the fact that duality interchanges inverse image and exceptional inverse image (see [DK19, Proposition 3.3.3]). Moreover, in the second and fourth isomorphism we have applied [DK19, Lemma 2.7.6].

Note that, except for the second line, all the isomorphisms require $\mathbb{R}$-constructibility, so it is essential here that K is $\mathbb{R}$-constructible and that direct and inverse images preserve this $\mathbb{R}$-constructibility (see [DK19, Proposition 3.3.3]). □

By restricting the moderate growth and rapid decay functors to the boundary of the real blow-up, we immediately obtain as a consequence a duality between the functors $\psi_f^{\leq 0}$ and $\psi_f^{\geq 0}$ (see Definition 4.5), up to a shift in the derived category $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(k_{\partial\widetilde{X}_f})$.

Corollary 5.5 (Local duality pairing on $\partial\widetilde{X}_f$)**.** *For* $K\in\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}k_X)$*, there is a canonical isomorphism*

$$\mathrm{D}_{\partial\widetilde{X}_f}(\psi_f^{\leq 0}K)\simeq\psi_f^{\geq 0}(\mathrm{D}^{\mathrm{E}}_XK)[1],$$

so that the associated pairing

$$\psi_f^{\leq 0}(K)\otimes\psi_f^{\geq 0}(\mathrm{D}^{\mathrm{E}}_XK)[1]\longrightarrow\omega_{\partial\widetilde{X}_f}$$

is perfect.

Proof. This is a similarly straightforward computation, using in addition the fact that Verdier dualizing interchanges i^{-1} and $i^!$. □

5.4. **The global duality pairings.** Applying (derived) global sections to the objects in Proposition 5.4 yields further duality statements that we can explore.

Let $K\in\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}k_X)$ be an $\mathbb{R}$-construcible enhanced ind-sheaf.

Specifically, the isomorphism $K^{\mathrm{mod}\,f}\simeq D_{\widetilde{X}_f}(\mathrm{D}^{\mathrm{E}}_XK)^{\mathrm{rd}\,f}$ implies the isomorphism

$$\begin{aligned}\mathrm{R}\Gamma(\widetilde{X}_f;K^{\mathrm{mod}\,f}) &\xrightarrow{\sim}\mathrm{RHom}(\mathrm{R}\Gamma_c(\widetilde{X}_f;(\mathrm{D}^{\mathrm{E}}_XK)^{\mathrm{rd}\,f}),k)\\ &=:\mathrm{R}\Gamma_c\big(\widetilde{X}_f;(\mathrm{D}^{\mathrm{E}}_XK)^{\mathrm{rd}\,f}\big)^\vee\end{aligned}\tag{11}$$

of bounded complexes of finite-dimensional k-vector spaces. Likewise,

$$\mathrm{D}_{\widetilde{X}_f}\big(K^{\mathrm{mod}\,f}\big)\simeq(\mathrm{D}^{\mathrm{E}}_XK)^{\mathrm{rd}\,f}$$

implies

$$\mathrm{R}\Gamma(\widetilde{X}_f;(\mathrm{D}^{\mathrm{E}}_XK)^{\mathrm{rd}\,f})\xrightarrow{\sim}\mathrm{R}\Gamma_c(\widetilde{X}_f;K^{\mathrm{mod}\,f})^\vee.$$

Either of these isomorphisms is equivalent to the following result.

Corollary 5.6. *The pairing*

$$\mathbb{H}^\ell(\widetilde{X}_f; K^{\mathrm{mod}\, f}) \otimes \mathbb{H}_c^{-\ell}(\widetilde{X}_f; (\mathrm{D}_X^{\mathrm{E}} K)^{\mathrm{rd}\, f}) \longrightarrow k$$

is perfect for all $K \in \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}k_X)$ *and* $\ell \in \mathbb{Z}$.

Proof. This follows directly from Proposition 5.3 and the above local duality, Proposition 5.4. □

Likewise, Proposition 5.3 and Corollary 5.5 imply the following result.

Corollary 5.7. *The pairing*

$$\mathbb{H}^\ell(\partial\widetilde{X}_f; \psi_f^{\leq 0} K) \otimes \mathbb{H}_c^{-\ell+1}(\partial\widetilde{X}_f; \psi_f^{\geq 0}\, \mathrm{D}_X^{\mathrm{E}} K) \longrightarrow \mathbb{H}_c^0(\partial\widetilde{X}_f; \omega_{\partial\widetilde{X}_f}) \simeq k$$

is perfect for all $K \in \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}k_X)$ *and* $\ell \in \mathbb{Z}$.

6. Duality between de Rham functors with growth conditions

6.1. **A duality of Kashiwara–Schapira.** In this section, we briefly recall a duality statement between tempered and Whitney holomorphic functions proved by M. Kashiwara and P. Schapira in [KS96] and [KS16].

It is well-known that the space of distributions is the topological dual of the space of compactly supported smooth functions (often called *test functions*). It is important to note that these spaces are infinite-dimensional complex vector spaces and this duality is really a duality of *topological* vector spaces, and not a duality in the category of complex vector spaces. The authors of [KS96] use the language of Fréchet nuclear spaces (vector spaces of type FN) and duals of Fréchet nuclear spaces (vector spaces of type DFN) to formulate their results (see [Gro55] for these notions, as well as [RR74] and the references therein).

The notions of tempered distributions and Whitney functions were studied in [KS96] and they are closely related to the notions of moderate growth and rapid decay (cf., e.g., [DK16, Proposition 7.2.10, Lemma 5.1.4]). On a real analytic manifold X, recall the functor $\bullet \overset{\mathrm{w}}{\otimes} \mathscr{C}_X^\infty \colon \mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_X) \to \mathrm{Mod}(\mathcal{D}_X)$ from loc. cit. as well as the ind-sheaf of tempered distributions $\mathcal{D}b_X^{\mathrm{t}}$ from [KS01]. The ind-sheaf $\mathcal{D}b_X^{\mathrm{t}\vee}$ is the ind-sheaf of tempered distribution densities on X (see, e.g., [KS16]). In [KS96], the authors proved the following duality result for these spaces (formulated here in the more modern notation of [KS01] and [KS16]).

Proposition 6.1 ([KS96, Proposition 2.2])**.** *Let X be a real analytic manifold and* $F \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$. *Then there exist natural topologies of type* FN *on* $\Gamma(X; F \overset{\mathrm{w}}{\otimes} \mathscr{C}_X^\infty)$ *and of type* DFN *on* $\Gamma_{\mathrm{c}}(X; \alpha_X \mathrm{R}\mathcal{I}hom(F, \mathcal{D}b_X^{\mathrm{t}\vee}))$ *and they are dual to each other.*

This result is then used to deduce the following duality in the context of holomorphic solutions of a coherent D-module on a complex manifold. We note that the transition from $\mathscr{C}_X^\infty$ (resp. $\mathcal{D}b_X^{\mathrm{t}\vee}$ to $\mathcal{O}_X$ (resp. Ω_X) amounts to taking Dolbeault complexes with coefficients in those objects (see [KS96, §5]).

Theorem 6.2 ([KS96, Theorem 6.1])**.** *Let X be a complex manifold of dimension* d_X, $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ *and* $F, G \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$. *Then the spaces* $\mathrm{R}\Gamma(X; \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M} \otimes G, F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X))$ *and* $\mathrm{R}\Gamma_{\mathrm{c}}(X; \alpha_X \mathrm{R}\mathcal{I}hom(F, \Omega_X^{\mathrm{t}})[d_X] \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X} (\mathcal{M} \otimes G))$ *are objects of* $\mathrm{D}^{\mathrm{b}}(\mathrm{FN})$ *and* $\mathrm{D}^{\mathrm{b}}(\mathrm{DFN})$*, respectively, and are dual to each other, functorially in* $\mathcal{M}$*,* F *and* G.

Later, in [KS16], this global result was extended to the following local statement in the category of $\mathbb{R}$-constructible sheaves.

Theorem 6.3 (see [KS16, Theorem 2.5.13])**.** *Let X be a complex manifold of dimension d_X, $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ and $F, G \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$. Then the two objects $\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X)$ and $\alpha_X \mathrm{R}\mathcal{I}hom(F, \Omega^{\mathrm{t}}_X)[d_X] \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X} \mathcal{M}$ are dual to each other in the category $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$.*

Let us remark that the key in deriving this result was the proof of the $\mathbb{R}$-constructibility of these two objects, which was derived from the constructiblity of enhanced solutions established in [DK16]. In particular, this allows the authors of [KS16] to forget the topology, so this duality is really a duality of sheaves of complex vector spaces.

6.2. **A conjecture of Sabbah.** In [Sab21], C. Sabbah conjectured the following duality statement.

Conjecture 6.4 (cf. [Sab21, Conjecture 4.13])**.** *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Then there is an isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\widetilde{X}_f})$*

$$\mathrm{D}_{\widetilde{X}_f} \mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}_f}(\mathcal{M}) \simeq \mathrm{DR}^{\mathrm{rd}}_{\widetilde{X}_f}(\mathbb{D}_X \mathcal{M}).$$

In the rest of this subsection, we will indicate a proof of this conjecture in the case of a normal crossing divisor using arguments very similar to the ones of Kashiwara–Schapira.

We will first prove the following variant on the real blow-up of X along a normal crossing divisor.

Proposition 6.5. *Let $D \subset X$ be a simple normal crossing divisor. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Then there is an isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\widetilde{X}_D})$*

$$\mathrm{D}_{\widetilde{X}_D} \mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}_D}(\mathcal{M}) \simeq \mathrm{DR}^{\mathrm{rd}}_{\widetilde{X}_D}(\mathbb{D}_X \mathcal{M}).$$

Let us first rewrite the objects in the statement of Proposition 6.5 in terms of tempered and Whitney holomorphic functions, as studied in [KS96], for example.

In the following, we will just write $\widetilde{X}$, ϖ for $\widetilde{X}_D$, ϖ_D. Let $\widetilde{X}^{\mathrm{tot}}_D$ be the total blowup of X along a normal crossing divisor, and let $\widetilde{X} \subset \widetilde{X}^{\mathrm{tot}}_D$ be the real blowup.

For the sheaf of moderate growth holomorphic function, we have

$$\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}} \simeq \alpha_{\widetilde{X}} \mathcal{O}^{\mathrm{t}}_{\widetilde{X}} \simeq \alpha_{\widetilde{X}} \varpi^! \mathrm{R}\mathcal{I}hom(\mathbb{C}_{X^*}, \mathcal{O}^{\mathrm{t}}_X)$$

and hence the moderate de Rham complex of a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is

$$\begin{aligned} \mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}}(\mathcal{M}) &\simeq (\varpi^{-1}\Omega_X \otimes_{\varpi^{-1}\mathcal{O}_X} \mathcal{A}^{\mathrm{mod}}_{\widetilde{X}}) \overset{\mathrm{L}}{\otimes}_{\varpi^{-1}\mathcal{D}_X} \varpi^{-1}\mathcal{M} \\ &\simeq \alpha_{\widetilde{X}} \varpi^! \mathrm{R}\mathcal{I}hom(\mathbb{C}_{X^*}, \Omega^{\mathrm{t}}_X) \overset{\mathrm{L}}{\otimes}_{\varpi^{-1}\mathcal{D}_X} \varpi^{-1}\mathcal{M} \\ &\simeq \alpha_{\widetilde{X}} \Omega^{\mathrm{t}}_{\widetilde{X}} \overset{\mathrm{L}}{\otimes}_{\varpi^{-1}\mathcal{D}_X} \varpi^{-1}\mathcal{M} \end{aligned} \tag{12}$$

On the other hand, the sheaf of smooth functions on $\widetilde{X}$ with rapid decay at $\partial\widetilde{X}$ is

$$\mathscr{C}^{\infty,\mathrm{rd}}_{\widetilde{X}} := (\mathbb{C}_{X^*} \overset{\mathrm{w}}{\otimes} \mathscr{C}^{\infty}_{\widetilde{X}^{\mathrm{tot}}})\big|_{\widetilde{X}}$$

and the sheaf of holomorphic function with rapid decay at the boundary is then

$$\mathcal{A}^{\mathrm{rd}}_{\widetilde{X}} := \mathrm{R}\mathcal{H}om_{\varpi^{-1}\mathcal{D}_{\overline{X}}}(\varpi^{-1}\mathcal{O}_{\overline{X}}, \mathscr{C}^{\infty,\mathrm{rd}}_{\widetilde{X}}).$$

Hence, the rapid decay de Rham complex of the dual of $\mathcal{M}$ is

$$\begin{aligned}\mathrm{DR}^{\mathrm{rd}}_{\widetilde{X}}(\mathbb{D}_X\mathcal{M}) &\simeq (\varpi^{-1}\Omega_X \otimes_{\varpi^{-1}\mathcal{O}_X} \mathcal{A}^{\mathrm{rd}}_{\widetilde{X}}) \overset{\mathrm{L}}{\otimes}_{\varpi^{-1}\mathcal{D}_X} \varpi^{-1}\mathbb{D}_X\mathcal{M}\\ &\simeq \mathrm{R}\mathcal{H}om_{\varpi^{-1}\mathcal{D}_X}(\varpi^{-1}\mathcal{M}, \mathcal{A}^{\mathrm{rd}}_{\widetilde{X}})[d_X] \end{aligned} \tag{13}$$

In order to prove Proposition 6.5, the aim is to get a duality between (12) and (13), and this reminds us of the duality in [KS16, (2.5.12), Theorem 2.5.13]. However, our statement is a duality on the real blow-up and it does not follow directly from loc. cit., but our proof will proceed along the same lines.

We will first construct a pairing between the two objects (12) and (13).

A pairing on $\widetilde{X}^{\mathrm{tot}}_D$. Since $\widetilde{X}^{\mathrm{tot}}_D$ is a real analytic manifold, we get from [KS16, (2.5.11)] a pairing

$$(\mathbb{C}_{X^*} \overset{\mathrm{w}}{\otimes} \mathscr{C}^{\infty}_{\widetilde{X}^{\mathrm{tot}}_D}) \otimes \alpha_{\widetilde{X}^{\mathrm{tot}}_D} \mathrm{R}\mathcal{I}hom(\mathbb{C}_{X^*}, \mathcal{D}b^{\mathrm{t}\vee}_{\widetilde{X}^{\mathrm{tot}}_D}) \to \omega_{\widetilde{X}^{\mathrm{tot}}_D}. \tag{14}$$

The induced pairing on $\widetilde{X}$. Equation (14) is equivalent to a morphism

$$\alpha_{\widetilde{X}^{\mathrm{tot}}_D} \mathrm{R}\mathcal{I}hom(\mathbb{C}_{X^*}, \mathcal{D}b^{\mathrm{t}\vee}_{\widetilde{X}^{\mathrm{tot}}_D}) \to \mathrm{D}_{\widetilde{X}^{\mathrm{tot}}_D}(\mathbb{C}_{X^*} \overset{\mathrm{w}}{\otimes} \mathscr{C}^{\infty}_{\widetilde{X}^{\mathrm{tot}}_D}).$$

Apply the exceptional inverse image along the (closed) embedding $i^{\mathrm{tot}}\colon \widetilde{X} \hookrightarrow \widetilde{X}^{\mathrm{tot}}_D$ to obtain

$$(i^{\mathrm{tot}})^! \alpha_{\widetilde{X}^{\mathrm{tot}}_D} \mathrm{R}\mathcal{I}hom(\mathbb{C}_{X^*}, \mathcal{D}b^{\mathrm{t}\vee}_{\widetilde{X}^{\mathrm{tot}}_D}) \to \mathrm{D}_{\widetilde{X}}(i^{\mathrm{tot}})^{-1}(\mathbb{C}_{X^*} \overset{\mathrm{w}}{\otimes} \mathscr{C}^{\infty}_{\widetilde{X}^{\mathrm{tot}}_D}). \tag{15}$$

Now observe that the left-hand side can be manipulated as follows:

$$\begin{aligned}(i^{\mathrm{tot}})^! &\alpha_{\widetilde{X}^{\mathrm{tot}}_D} \mathrm{R}\mathcal{I}hom(\mathbb{C}_{X^*}, \mathcal{D}b^{\mathrm{t}\vee}_{\widetilde{X}^{\mathrm{tot}}_D})\\ &\simeq (i^{\mathrm{tot}})^{-1}\mathrm{R}\mathcal{H}om(\mathbb{C}_{\widetilde{X}}, \alpha_{\widetilde{X}^{\mathrm{tot}}_D}\mathrm{R}\mathcal{I}hom(\mathbb{C}_{X^*}, \mathcal{D}b^{\mathrm{t}\vee}_{\widetilde{X}^{\mathrm{tot}}_D}))\\ &\simeq (i^{\mathrm{tot}})^{-1}\mathrm{R}\mathcal{H}om(\beta_{\widetilde{X}^{\mathrm{tot}}_D}\mathbb{C}_{\widetilde{X}}, \mathrm{R}\mathcal{I}hom(\mathbb{C}_{X^*}, \mathcal{D}b^{\mathrm{t}\vee}_{\widetilde{X}^{\mathrm{tot}}_D}))\\ &\simeq (i^{\mathrm{tot}})^{-1}\alpha_{\widetilde{X}^{\mathrm{tot}}_D}\mathrm{R}\mathcal{I}hom(\beta_{\widetilde{X}^{\mathrm{tot}}_D}\mathbb{C}_{\widetilde{X}} \otimes \iota_{\widetilde{X}^{\mathrm{tot}}_D}\mathbb{C}_{X^*}, \mathcal{D}b^{\mathrm{t}\vee}_{\widetilde{X}^{\mathrm{tot}}_D})\\ &\simeq \alpha_{\widetilde{X}^{\mathrm{tot}}_D}(i^{\mathrm{tot}})^{-1}\mathrm{R}\mathcal{I}hom(\mathbb{C}_{X^*}, \mathcal{D}b^{\mathrm{t}\vee}_{\widetilde{X}^{\mathrm{tot}}_D}) \end{aligned} \tag{16}$$

Here, the second isomorphism follows from [KS01, Proposition 5.1.10], and the last one from the fact that $\beta_{\widetilde{X}^{\mathrm{tot}}_D}\mathbb{C}_{\widetilde{X}} \otimes \iota_{\widetilde{X}^{\mathrm{tot}}_D}\mathbb{C}_{X^*} \simeq \iota_{\widetilde{X}^{\mathrm{tot}}_D}\mathbb{C}_{X^*}$ (see [KS01] and also [Ho22, Proof of Lemma 3.2]).

Next, let us prove the following result similar to [KS16, Lemma 2.5.7].

Lemma 6.6. *Set $Y := (\varpi^{\mathrm{tot}})^{-1}(X^*)$. Then there is an isomorphism*

$$\mathrm{R}\mathcal{I}hom(\mathbb{C}_Y, \mathcal{D}b^{\mathrm{t}\vee}_{\widetilde{X}^{\mathrm{tot}}_D}) \simeq (\varpi^{\mathrm{tot}})^!\mathrm{R}\mathcal{I}hom(\mathbb{C}_{X^*}, \mathcal{D}b^{\mathrm{t}\vee}_X). \tag{17}$$

Proof. Note that, locally, Y is the disjoint union of 2^r connected components (r being the number of smooth components of the normal crossing divisor D), each homeomorphically mapped to X^* via ϖ^{tot}. Let us write $Y = \bigsqcup_{\nu\in\{+,-\}^r} X^* \times \{\nu\}$. (Recall that $X^* \times \{+,\dots,+\}$ is the component canonically identified with X^* in

$\widetilde{X} \subset \widetilde{X}^{\mathrm{tot}}_D$.) Then we have

$$(\varpi^{\mathrm{tot}})^! \mathrm{R}\mathcal{I}hom(\mathbb{C}_{X^*}, \mathcal{D}b^{\mathrm{t}\vee}_X) \simeq \mathrm{R}\mathcal{I}hom\big((\varpi^{\mathrm{tot}})^{-1}\mathbb{C}_{X^*}, (\varpi^{\mathrm{tot}})^! \mathcal{D}b^{\mathrm{t}\vee}_X\big)$$
$$\simeq \bigoplus_{\nu \in \{+,-\}^r} \mathrm{R}\mathcal{I}hom\big(\mathbb{C}_{X^* \times \{\nu\}}, (\varpi^{\mathrm{tot}})^! \mathcal{D}b^{\mathrm{t}\vee}_X\big).$$

Now, by [KS16, Theorem 2.5.6], one has $(\varpi^{\mathrm{tot}})^! \mathcal{D}b^{\mathrm{t}\vee}_X \simeq \mathcal{D}b^{\mathrm{t}\vee}_{\widetilde{X}^{\mathrm{tot}}_D} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_{\widetilde{X}^{\mathrm{tot}}_D}} \mathcal{D}_{\widetilde{X}^{\mathrm{tot}}_D \to X}$, and since $\mathcal{D}_{\widetilde{X}^{\mathrm{tot}}_D} \to \mathcal{D}_{\widetilde{X}^{\mathrm{tot}}_D \to X}$ is an isomorphism on each $X^* \times \{\nu\}$, we obtain the desired result. □

Applying the functor $(i^{\mathrm{tot}})^!$ to the isomorphism (17), the left-hand side is the following:

$$(i^{\mathrm{tot}})^! \mathrm{R}\mathcal{I}hom(\mathbb{C}_Y, \mathcal{D}b^{\mathrm{t}\vee}_{\widetilde{X}^{\mathrm{tot}}_D}) \simeq (i^{\mathrm{tot}})^{-1} \mathrm{R}\mathcal{I}hom(\mathbb{C}_{\widetilde{X}}, \mathrm{R}\mathcal{I}hom(\mathbb{C}_Y, \mathcal{D}b^{\mathrm{t}\vee}_{\widetilde{X}^{\mathrm{tot}}_D}))$$
$$\simeq (i^{\mathrm{tot}})^{-1} \mathrm{R}\mathcal{I}hom(\mathbb{C}_{\widetilde{X}} \otimes \mathbb{C}_Y, \mathcal{D}b^{\mathrm{t}\vee}_{\widetilde{X}^{\mathrm{tot}}_D})$$
$$\simeq (i^{\mathrm{tot}})^{-1} \mathrm{R}\mathcal{I}hom(\mathbb{C}_{X^*}, \mathcal{D}b^{\mathrm{t}\vee}_{\widetilde{X}^{\mathrm{tot}}_D})$$

Hence, taking together the isomorphism (17) with our computation (16), we obtain

$$(i^{\mathrm{tot}})^! \alpha_{\widetilde{X}^{\mathrm{tot}}_D} \mathrm{R}\mathcal{I}hom(\mathbb{C}_{X^*}, \mathcal{D}b^{\mathrm{t}\vee}_{\widetilde{X}^{\mathrm{tot}}_D}) \simeq \alpha_{\widetilde{X}} \varpi^! \mathrm{R}\mathcal{I}hom(\mathbb{C}_{X^*}, \mathcal{D}b^{\mathrm{t}\vee}_X) \tag{18}$$

and hence the morphism (15) yields a pairing

$$(i^{\mathrm{tot}})^{-1}(\mathbb{C}_{X^*} \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_{\widetilde{X}^{\mathrm{tot}}_D}) \otimes \alpha_{\widetilde{X}} \varpi^! \mathrm{R}\mathcal{I}hom(\mathbb{C}_{X^*}, \mathcal{D}b^{\mathrm{t}\vee}_X) \to \omega_{\widetilde{X}}. \tag{19}$$

The pairing for the de Rham functors. Making the transition to Dolbeault complexes (i.e., going to the holomorphic setting) and adding a D-module, the above pairing will induce a pairing

$$\mathrm{DR}^{\mathrm{rd}}_{\widetilde{X}}(\mathbb{D}_X \mathcal{M}) \otimes \mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}}(\mathcal{M}) \to \omega_{\widetilde{X}} \tag{20}$$

(just as (2.5.11) induces (2.6.12) in [KS16]).

Perfectness of the pairing. We first prove an analogue (indeed, a corollary) of [KS96, Proposition 2.2], which will be the key to the proof of the perfectness.

Proposition 6.7. *There exist natural topologies of type* FN *on* $\Gamma(\widetilde{X}; (i^{\mathrm{tot}})^{-1}(\mathbb{C}_{X^*} \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_{\widetilde{X}}))$ *and of type* DFN *on* $\Gamma_c(\widetilde{X}; (i^{\mathrm{tot}})^{-1} \mathrm{R}\mathcal{H}om(\mathbb{C}_{X^*}, \mathcal{D}b^{\mathrm{t}\vee}_{\widetilde{X}^{\mathrm{tot}}_D}))$ *and the two spaces are dual to each other with respect to these topologies.*

Proof. Since $\widetilde{X}^{\mathrm{tot}}_D$ is a real analytic manifold, [KS96, Proposition 2.2] gives us a (topological) duality between the FN space $\Gamma(\widetilde{X}^{\mathrm{tot}}_D; \mathbb{C}_{X^*} \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_{\widetilde{X}^{\mathrm{tot}}_D})$ and the DFN space $\Gamma_c(\widetilde{X}^{\mathrm{tot}}_D; \mathrm{R}\mathcal{I}hom(\mathbb{C}_{X^*}, \mathcal{D}b^{\mathrm{t}\vee}_{\widetilde{X}^{\mathrm{tot}}_D}))$. To conclude, we observe that both $\mathbb{C}_{X^*} \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_{\widetilde{X}^{\mathrm{tot}}_D}$ and $\mathrm{R}\mathcal{I}hom(\mathbb{C}_{X^*}, \mathcal{D}b^{\mathrm{t}\vee}_{\widetilde{X}^{\mathrm{tot}}_D})$ are supported on the closed subspace $\widetilde{X} \subset \widetilde{X}^{\mathrm{tot}}_D$, and hence

$$\Gamma\big(\widetilde{X}; (i^{\mathrm{tot}})^{-1}(\mathbb{C}_{X^*} \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_{\widetilde{X}^{\mathrm{tot}}_D})\big) \simeq \Gamma\big(X; \mathbb{C}_{X^*} \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_{\widetilde{X}^{\mathrm{tot}}_D}\big)$$

and

$$\Gamma_c\big(\widetilde{X}; (i^{\mathrm{tot}})^{-1} \mathrm{R}\mathcal{H}om(\mathbb{C}_{X^*}, \mathcal{D}b^{\mathrm{t}\vee}_{\widetilde{X}^{\mathrm{tot}}_D})\big) \simeq \Gamma_c\big(X; \mathrm{R}\mathcal{I}hom(\mathbb{C}_{X^*}, \mathcal{D}b^{\mathrm{t}\vee}_{\widetilde{X}^{\mathrm{tot}}_D})\big).$$

□

This implies (going to the holomorphic setting similarly to [KS96, Proposition 5.2]) a duality between the objects

$$\mathrm{R}\Gamma(X;\mathcal{A}^{\mathrm{rd}}_{\widetilde{X}}) \in \mathrm{D}^{\mathrm{b}}(\mathrm{FN})$$

and

$$\mathrm{R}\Gamma_c(X;\alpha_{\widetilde{X}}\Omega^{\mathrm{t}}_{\widetilde{X}}) \in \mathrm{D}^{\mathrm{b}}(\mathrm{DFN}).$$

We can then use the technique of [KS96, Theorem 6.1 and Appendix] to prove the following.

Proposition 6.8. *Let $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ and let $G \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_{\widetilde{X}})$. Then we can define*

$$\mathrm{R}\Gamma\big(X;\mathrm{R}\mathcal{H}om_{\varpi^{-1}\mathcal{D}_X}(\varpi^{-1}\mathcal{M}\otimes G,\mathcal{A}^{\mathrm{rd}}_{\widetilde{X}})\big) \simeq \mathrm{RHom}\big(G,\mathrm{DR}^{\mathrm{rd}}_{\widetilde{X}}(\mathbb{D}_X\mathcal{M})\big) \in \mathrm{D}^{\mathrm{b}}(\mathrm{FN})$$

and

$$\mathrm{R}\Gamma_c\big(X;\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}}(\mathcal{M})\otimes G\big) \in \mathrm{D}^{\mathrm{b}}(\mathrm{DFN})$$

and these are dual to each other.

Proof. By the methods developed in loc. cit. (whose details we will not elaborate on here, see the proof of [KS96, Theorem 6.1]), the proof can be reduced to the case $\mathcal{M} = (\mathcal{D}_X)_U$ for some relatively compact open $U \subset X$ and $G = \mathbb{C}_V$ for some relatively compact open subanalytic $V \subset \widetilde{X}$, and this reduces the statement to Proposition 6.7 and its holomorphic analog. □

The final step in proving Proposition 6.5 is now the following lemma, which is analogous to the idea of [KS16, Lemma 2.5.12].

Lemma 6.9. *If one of the objects $\mathrm{DR}^{\mathrm{rd}}_{\widetilde{X}}(\mathbb{D}_X\mathcal{M})$ or $\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}}(\mathcal{M})$ is $\mathbb{R}$-constructible, then the pairing (20) is perfect and the objects $\mathrm{DR}^{\mathrm{rd}}_{\widetilde{X}}(\mathbb{D}_X\mathcal{M})$ and $\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}}(\mathcal{M})$ are dual to each other in the category $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_{\widetilde{X}})$. In particular, the other object is also $\mathbb{R}$-constructible.*

Proof. Assume that $\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}}(\mathcal{M})$ is $\mathbb{R}$-constructible. Apply the functor $\mathrm{R}\Gamma_c(U;\bullet)$ for any relatively compact open subanalytic $U \subset \widetilde{X}$ to the morphism

$$\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}}(\mathcal{M}) \to \mathrm{D}_{\widetilde{X}}\mathrm{DR}^{\mathrm{rd}}_{\widetilde{X}}(\mathbb{D}_X\mathcal{M})$$

coming from the pairing (20) to obtain

$$\mathrm{R}\Gamma(U;\mathrm{DR}^{\mathrm{rd}}_{\widetilde{X}}(\mathbb{D}_X\mathcal{M})) \to \mathrm{R}\Gamma_c(U;\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}}(\mathcal{M}))^*.$$

Since $\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}}(\mathcal{M})$ was assumed to be $\mathbb{R}$-constructible, we know (see [KS90, Corollary 8.4.11]) that $\mathrm{R}\Gamma_c(U;\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}}(\mathcal{M}))$ is a perfect complex and therefore has, in particular, finite-dimensional cohomologies, and hence the topological dual and the usual dual as a vector space coincide. This morphism is an isomorphism by Proposition 6.8, which concludes the proof.

The case where $\mathrm{DR}^{\mathrm{rd}}_{\widetilde{X}}(\mathcal{M})$ is $\mathbb{R}$-constructible is similar. □

We can now finally prove the above duality statement in the case of a normal crossing divisor.

Proof of Proposition 6.5. By Lemma 6.9, we are reduced to proving that the complex $\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}}(\mathcal{M})$ is $\mathbb{R}$-constructible. This is, however, clear by (a version on $\widetilde{X}_D$ of) Lemma 5.1 since we have already proved $\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}_D}(\mathcal{M}) \simeq \mathsf{sh}_{\widetilde{X}_D}\big(\mathrm{E}\tilde{j}_{D*}\mathrm{E}j^{-1}\mathrm{DR}^{\mathrm{E}}_X(\mathcal{M})\big)$

(see Proposition 3.1) and we know that $\mathrm{DR}_X^{\mathrm{E}}(\mathcal{M})$ is $\mathbb{R}$-constructible (see [DK16, Theorem 9.3.2]). This concludes the proof. □

Corollary 6.10. *Let $f\colon X \to \mathbb{C}$ be a holomorphic function such that $D := f^{-1}(0)$ is a simple normal crossing divisor. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Then there is an isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\widetilde{X}_f})$*

$$\mathrm{D}_{\widetilde{X}_f}\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}_f}(\mathcal{M}) \simeq \mathrm{DR}^{\mathrm{rd}}_{\widetilde{X}_f}(\mathbb{D}_X\mathcal{M}).$$

Proof. This follows directly from Proposition 6.5 by applying the direct image functor along the natural morphism $\varpi_{D,f}\colon \widetilde{X}_D \to \widetilde{X}_f$. We note that this morphism is proper and hence self-dual, and that this direct image turns the moderate (resp. rapid decay) de Rham complex on $\widetilde{X}_D$ into the moderate (resp. rapid decay) de Rham complex on $\widetilde{X}_f$ (see [Moc14, Proposition 4.7.4]). □

With this result in hand, we can now derive the statement motivated in (5) in the case of a normal crossing divisor.

Corollary 6.11. *If $D \subset X$ is a simple normal crossing divisor, we have an isomorphism*

$$\mathrm{DR}^{\mathrm{rd}}_{\widetilde{X}_D}(\mathbb{D}_X\mathcal{M}) \simeq \mathsf{sh}_{\widetilde{X}_D}\big(\mathrm{E}\tilde{j}_{D!!}\mathrm{E}j^{-1}\mathrm{DR}^{\mathrm{E}}_X(\mathbb{D}_X\mathcal{M})\big).$$

If $f\colon X \to \mathbb{C}$ is a holomorphic function such that $f^{-1}(0)$ is a simple normal crossing divisor, we have an isomorphism

$$\mathrm{DR}^{\mathrm{rd}}_{\widetilde{X}_f}(\mathbb{D}_X\mathcal{M}) \simeq \mathsf{sh}_{\widetilde{X}_f}\big(\mathrm{E}\tilde{j}_{f!!}\mathrm{E}j^{-1}\mathrm{DR}^{\mathrm{E}}_X(\mathbb{D}_X\mathcal{M})\big).$$

Proof. The proofs of the two statements are indentical, so let us give the argument for $\widetilde{X}_D$.

Recall (from Proposition 3.1) that

$$\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}_D}(\mathcal{M}) \simeq \mathsf{sh}_{\widetilde{X}_D}(\mathrm{E}\tilde{j}_{D*}\mathrm{E}j^{-1}\mathrm{DR}^{\mathrm{E}}_X(\mathcal{M})).$$

Then the assertion of the corollary is just the dual of this isomorphism: The left-hand sides are dual to each other by Proposition 6.5 just proved. For the right-hand sides, we see that

$$\begin{aligned}\mathsf{sh}_{\widetilde{X}_D}(\mathrm{E}\tilde{j}_{D*}\mathrm{E}j^{-1}\mathrm{DR}^{\mathrm{E}}_X(\mathbb{D}_X\mathcal{M})) &\simeq \mathsf{sh}_{\widetilde{X}_D}(\mathrm{E}\tilde{j}_{D*}\mathrm{E}j^{-1}\mathrm{D}^{\mathrm{E}}_X\mathrm{DR}^{\mathrm{E}}_X(\mathbb{D}_X\mathcal{M}))\\ &\simeq \mathsf{sh}_{\widetilde{X}_D}(\mathrm{D}^{\mathrm{E}}_{\widetilde{X}_D}\mathrm{E}\tilde{j}_{D!!}\mathrm{E}j^{!}\mathrm{DR}^{\mathrm{E}}_X(\mathcal{M}))\\ &\simeq \mathrm{D}_{\widetilde{X}_D}\mathsf{sh}_{\widetilde{X}_D}(\mathrm{E}\tilde{j}_{D!!}\mathrm{E}j^{-1}\mathrm{DR}^{\mathrm{E}}_X(\mathcal{M})\end{aligned}$$

since duality commutes with the enhanced de Rham functor (see [DK16, Theorem 9.4.8] and the sheafification functor (see [DK21b, Proposition 3.13]), and duality interchanges $\mathrm{E}\tilde{j}_{D*}$ with $\mathrm{E}\tilde{j}_{D!!}$ and $\mathrm{E}j^{-1}$ with $\mathrm{E}j^{!}$ (see [DK19, Proposition 3.3.3]). Note that $\mathrm{E}j^{-1} \simeq \mathrm{E}j^{!}$ since j is an open embedding. Since all the objects involved are $\mathbb{R}$-constructible (and hence duality is an involution on these objects), this concludes the proof. □

6.3. **Duality pairings for algebraic connections.** A well-known result of S. Bloch and H. Esnault (in dimension one, see [BE04]) and M. Hien (for dimensions greater than one, see [Hie09] and also [Hie07]) shows that, for a flat algebraic connection (E, ∇) on a smooth quasi-projective variety U, there is a pairing between algebraic de Rham cohomology with coefficients in (E, ∇) and rapid decay homology cycles, inducing a duality between the respective (co-)homology spaces (see 22 for the precise statement). Concretely, the pairing can be realized as an integration, or period, pairing of differential forms $\alpha \otimes e$ with coefficients in E and of chains $\gamma \otimes \epsilon^\vee$ with coefficients in $E^\vee$ having rapid decay along the boundary:

$$(\alpha \otimes e, \gamma \otimes \epsilon^\vee) \mapsto \int_\gamma \langle e, \epsilon^\vee \rangle \, \alpha \tag{21}$$

where $\langle e, \epsilon^\vee \rangle$ denotes the natural contraction $E \otimes_{\mathcal{O}_U} E^\vee \to \mathcal{O}_U$.

Proposition 6.12. *Let (E, ∇) be a flat (algebraic) connection on a smooth quasi-projective variety U over $\mathbb{C}$, and let $(E^\vee, \nabla^\vee)$ be the dual connection on U. Then there is a perfect pairing of finite-dimensional $\mathbb{C}$-vector spaces*

$$\mathrm{H}^\ell_{\mathrm{dR}}\big(U; (E, \nabla)\big) \otimes \mathrm{H}^{\mathrm{rd}}_\ell\big(U; (E^\vee, \nabla^\vee)\big) \to \mathbb{C} \tag{22}$$

where H_{dR} denotes algebraic de Rham cohomology, and H^{rd} denotes rapid decay homology (see [BE04] *and* [Hie09]*).*

Proof. Recall from [Hie09] that we can describe these de Rham cohomology groups and rapid decay homology groups as the hypercohomologies of complexes of sheaves on the real blow-up space: We have

$$\mathrm{H}^\ell_{\mathrm{dR}}\big(U; (E, \nabla)\big) \simeq \mathbb{H}^\ell\big(\widetilde{X}_D; \mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}_D}(\mathcal{M})\big)$$

(see [Sab00]) and

$$\mathrm{H}^{\mathrm{rd}}_\ell\big(U; (E^\vee, \nabla^\vee)\big) \simeq \mathbb{H}^{-\ell}\big(\widetilde{X}_D; \mathrm{DR}^{\mathrm{rd}}_{\widetilde{X}_D}(\mathcal{M}^\vee)\big).$$

Here, (X, D) is a good compactification of U, i.e., $D = X \setminus U$ has normal crossings and (E, ∇) admits a good formal structure with respect to (X, D). (The existence of such a good compatification was conjectured and proved in certain cases in [Sab00], and it was proved in general in [Moc09], [Moc11].) Furthermore, $\mathcal{M}$ is the (analytic) meromorphic connection associated to the algebraic connection (E, ∇). In other words, if $\mathcal{E}$ is the algebraic $\mathcal{D}_U$-module defined by the connection (E, ∇) and $j \colon U \hookrightarrow X$ is the inclusion, then $\mathcal{M} = (j_*\mathcal{E})^{\mathrm{an}}$ is the associated analytic $\mathcal{D}_X$-module and satisfies $\mathcal{M}(*D) \simeq \mathcal{M}$. Moreover, $\mathcal{M}^\vee$ denotes the dual meromorphic connection, which is the meromorphic connection associated to $(E^\vee, \nabla^\vee)$ (or, equivalently, $\mathcal{M}^\vee = (\mathbb{D}_X\mathcal{M})(*D)$).

With these identifications, the claim follows after noting

$$\mathrm{DR}^{\mathrm{mod}}_{\widetilde{X}}(\mathcal{M}) \simeq \big(\mathrm{DR}^{\mathrm{E}}_X(\mathcal{M})\big)^{\mathrm{mod}\, D},$$

and

$$\mathrm{DR}^{\mathrm{rd}}_{\widetilde{X}_D}(\mathcal{M}^\vee) \simeq \big(\mathrm{DR}^{\mathrm{E}}_X(\mathcal{M}^\vee)\big)^{\mathrm{rd}\, D} \simeq \big(\mathrm{DR}^{\mathrm{E}}_X(\mathbb{D}_X\mathcal{M})\big)^{\mathrm{rd}\, D} \simeq \big(\mathrm{D}^{\mathrm{E}}_X(\mathrm{DR}^{\mathrm{E}}_X(\mathcal{M}))\big)^{\mathrm{rd}\, D},$$

and then applying (a version on $\widetilde{X}_D$ of) Proposition 5.6. (Note that $\widetilde{X}_D$ is compact.) □

References

[AHV18] J. M. Aroca, H. Hironaka, J. L. Vicente, *Complex Analytic Desingularization*, Springer, Tokyo 2018.

[BE04] S. Bloch and H. Esnault, *Homology for irregular connections*, J. de Théorie des Nom. de Bordeaux **16-2** (2004), 357–371.

[Boa21] P. Boalch, *Topology of the Stokes phenomenon*, Proc. Symp. Pure Math **103** (2021), 55–100.

[DK16] A. D'Agnolo and M. Kashiwara, *Riemann-Hilbert correspondence for holonomic D-modules*, Publ. Math. Inst. Hautes Études Sci. **123** (2016), 69–197.
[DK18] A. D'Agnolo and M. Kashiwara, *A microlocal approach to the enhanced Fourier–Sato transform in dimension one*, Adv. Math. **339** (2018), 1–59.
[DK19] A. D'Agnolo and M. Kashiwara, *Enhanced perversities*, J. reine angew. Math. **751** (2019), 185–241.
[DK20] A. D'Agnolo and M. Kashiwara, *Enhanced nearby and vanishing cycles in dimension one and Fourier transform*, `arXiv:2002.11341v2` (2021), to appear in Publ. Res. Inst. Math. Sci.
[DK21a] A. D'Agnolo and M. Kashiwara, *Enhanced specialization and microlocalization*, Selecta Math. **27**, Article no. 7 (2021).
[DK21b] A. D'Agnolo and M. Kashiwara, *On a topological counterpart of regularization for holonomic $\mathcal{D}$-modules*, J. Éc. polytech. Math. **8** (2021), 27–55.
[FSY21] J. Fresán, C. Sabbah, and J.-D. Yu, *Quadratic relations between periods of connections*,`arXiv:2005.11525` (2021), to appear in Tohoku Math. J.
[Gro55] A. Grothendieck, *Produits Tensoriels Topologiques et Espaces Nucléaires, Mem. Amer. Math. Soc.* **16** *(1955).*
[Hie07] M. Hien, *Periods for irregular singular connections on surfaces*, Math. Ann. **337** (2007), 631–669.
[Hie09] M. Hien, *Periods for flat algebraic connections*, Invent. Math. **178** (2009), 1–22.
[Ho22] A. Hohl, *D-modules of pure Gaussian type and enhanced ind-sheaves*, Manuscripta Math. **167** (2022), 435-467.
[HTT08] R. Hotta, K. Takeuchi, and T. Tanisaki, *D-Modules, Perverse Sheaves, and Representation Theory*, Progress in Mathematics, vol. 236, Birkhäuser, Boston 2008.
[Ito20] Y. Ito, *$\mathbb{C}$-constructible enhanced ind-sheaves*, Tsukuba J. Math. **44** (2020), 155–201.
[IT20] Y. Ito and K. Takeuchi. *On Irregularities of Fourier Transforms of Regular Holonomic D-Modules*, Adv. Math. **366** (2020), 107093.
[Kas84] Masaki Kashiwara, *The Riemann-Hilbert problem for holonomic systems*, Publ. Res. Inst. Math. Sci. **20** (1984), 319–365.
[Kas03] Masaki Kashiwara, *D-modules and Microlocal Calculus*, Translations of Mathematical Monographs, vol. 217, American Mathematical Society, Providence, RI 2003.
[KS90] M. Kashiwara, P. Schapira, *Sheaves on Manifolds*, Grundlehren der mathematischen Wissenschaften **292**, Springer, Berlin 1990.
[KS96] M. Kashiwara, P. Schapira, *Moderate and formal cohomology associated with constructible sheaves*, Mem. Soc. Math. France **64** (1996).
[KS01] M. Kashiwara and P. Schapira, *Ind-sheaves*, Astérisque **271** (2001).
[KS06] M. Kashiwara and P. Schapira, *Categories and Sheaves*, Grundlehren der mathematischen Wissenschaften **332**, Springer, Berlin 2006.
[KS16] M. Kashiwara and P. Schapira, *Regular and irregular holonomic D-modules*, London Mathematical Society Lecture Note Series **433**, Cambridge University Press, Cambridge 2016.
[Kuw21] T. Kuwagaki, *Irregular Perverse sheaves*, Compositio Math. **157** (2021), 573–624.
[Mal91] B. Malgrange, *Équations Différentielles à Coefficients Polynomiaux*, Progress in Mathematics **96**, Birkhäuser, Boston 1991.
[Meb84] Z. Mebkhout, *Une autre équivalence de catégories*, Compos. Math. **51** (1984), 63–88.
[Moc09] T. Mochizuki, *Good formal structure for meromorphic flat connections on smooth projective surfaces*, Adv. Stud. Pure Math. **2009** (2009), 223–253.
[Moc11] T. Mochizuki, *Wild harmonic bundles and wild pure twistor D-modules*, Astérisque **340** Soc. Math. France, 2011.
[Moc14] T. Mochizuki, *Holonomic $\mathcal{D}$-modules with Betti structure*, Mém. Soc. Math. Fr. (N.S.) **139–140** (2014).
[Moc18] T. Mochizuki, *Curve test for enhanced ind-sheaves and holonomic D-modules*, Preprint, `arXiv:1610.08572v3` (2018).
[Sab00] C. Sabbah, *Équations différentielles à points singuliers irréguliers et phénomène de Stokes en dimension 2*, Astérisque **263**, Soc. Math. France, 2000.
[RR74] J.-P. Ramis and G. Ruget, *Residus et dualité*, Inventiones Math. **26** (1974), 89–131.
[Sab13] C. Sabbah, *Introduction to Stokes Structures*, Lecture Notes in Mathematics **2060**, Springer, Berlin 2013.

[Sab21] C. Sabbah, *Moderate and rapid-decay nearby cycles for holonomic $\mathscr{D}$-modules*, Preprint, https://perso.pages.math.cnrs.fr/users/claude.sabbah/articles/sabbah_nearby-stokes.pdf (2021).

[Siu70] Yum-Tong Siu, *Holomorphic functions of polynomial growth on bounded domains*, Duke Math. J. **37** (1970), 77–84.

(B.H.) University of Wisconsin-Madison, 480 Lincoln Drive, Madison, Wisconsin 53706, USA

Email address: bhepler@wisc.edu

(A.H.) Université Paris Cité and Sorbonne Université, CNRS, IMJ-PRG (Institut de Mathématiques de Jussieu-Paris Rive Gauche), 75013 Paris, France

Email address: andreas.hohl@imj-prg.fr