# RESEARCH STATEMENT

BRIAN HEPLER


## Summary

My research is centered around the theme of investigating the (micro)local topology of singular spaces using the machinery of various derived categories of sheaves, with a particular focus on vanishing cycles and perverse sheaves. Lately, this is manifested in two long-term projects. The first is a nearly 40-year-old open problem by Lê Dũng Tráng [198] concerning the equisingularity of germs of complex analytic surfaces $V(f) := \{f = 0\}$ in $\mathbb{C}^3$ with one-dimensional singular locus. The second topic concerns the microlocal data of irregular perverse sheaves arising from the proof of the irregular Riemann-Hilbert correspondence proved by Andrea D'Agnolo and Masaki Kashiwara in 2016 [D'A16], which I investigate using a new generalization of vanishing cycles [Hep21] designed to handle such objects.

Lê's Conjecture concerns the deep relationship between the topological and analytic properties of surface germs: if the link of $V(f)$ (the intersection of $V(f)$ with a sufficiently small sphere centered at 0) is homeomorphic to a 3-sphere, then the singular locus of $V(f)$ is a smooth curve at 0. The link of a surface is one of the most important pieces of data associated to a singularity, and this hypothesis places strict constraints on the local topology of $V(f)$ at 0–in particular, it implies that the normalization of $V(f)$ is smooth, and is a bijection. One pictures $V(f)$ as "folding up" $\mathbb{C}^2$, with the "creases" corresponding to the singular locus. My approach, together with Laurentiu Maxim, is via the machinery of Saito's mixed Hodge modules. I will give some motivation for this Conjecture, as well as an equivalent formulation in the language of mixed Hodge modules below in the research summary.

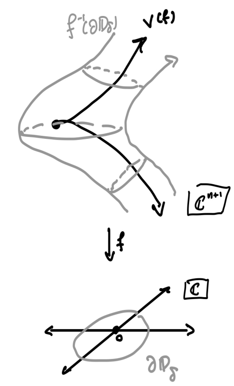
Defining the category of perverse sheaves on a complex manifold $X$ is never an easy task in such settings; the canonical joke is that they are neither perverse nor sheaves [Bei81]. In the simplest situations, a perverse sheaf is generically just a locally constant sheaf of finite rank away from some divisor in $X$. Irregular perverse sheaves are then, analogously, generically given by a local system on the complement of a divisor, equipped with a filtration that controls the "asymptotic behavior" of sections as one approaches the divisor from different directions. Making this analogy precise using generalizations of the microlocal sheaf theory of Kashiwara-Schapira [Kas90] has formed the bulk of my postdoctoral research at UW-Madison. Understanding such microlocal data associated to sheaves has become a central problem in geometric representation theory (via the geometric Langlands programme and the computation of Gopakumar-Vafa invariants), symplectic topology (via wrapped Fukaya categories and sheaf quantization), and mathematical physics (via branes in string theory and various formulations of quantum gravity). It is then a natural question to ask what new microlocal data this vast generalization of perverse sheaves can see, that classical sheaf theory cannot.

## 1. Lê's Conjecture via Mixed Hodge Modules

This project is the investigation of a classic conjecture (due to Lê [198]) in the field of singularities of complex analytic spaces, specifically on the so-called "equisingularity" of certain surfaces with non-isolated singularities inside $\mathbb{C}^3$. Let us briefly recall some of the essential notions from singularity theory.

The study of (the topology of) complex hypersurfaces with **isolated** singularities largely started with the foundational work of Milnor [Mil68]; the local, ambient topological type of a hypersurface $V(f) \subseteq \mathbb{C}^{n+1}$ at a singular point $p \in V(f)$ is completely determined by a fibration (called the Milnor fibration) defined on a "tube" around the hypersurface near $p$. More precisely, for $0 < \delta \ll \epsilon \ll 1$, the defining function $f$ restricts to a smooth, locally trivial fibration

$$\hat{f} : B_\epsilon(p) \cap f^{-1}(\partial \mathbb{D}_\delta) \to \partial \mathbb{D}_\delta$$



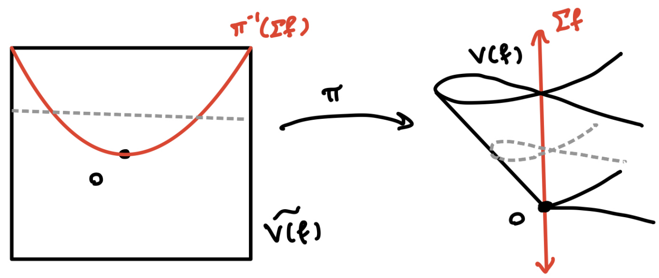

where $B_\epsilon(p)$ is an open ball of radius $\epsilon$ at $p$ in $\mathbb{C}^{n+1}$ (with respect to any Riemannian metric), and $\mathbb{D}_\delta$ is a disk of radius $\delta$ around 0 in $\mathbb{C}$. The fiber of $\hat{f}$ is called the **Milnor fiber** of $f$ at $p$, denoted $F_{f,p}$, is a compact, orientable manifold of dimension $2n$ that is homotopy equivalent to a finite bouquet of $n$-spheres. The number of such spheres is called the **Milnor number** of $f$ at $p$, and is denoted $\mu_p(f)$.

Milnor's fibration still exists in the more general context of hypersurfaces with non-isolated singularities, but the associated Milnor fiber is no longer as nice as in the isolated case. It still has the homotopy-type of a finite CW complex, and if the singular locus $\Sigma f$ has $\dim_0 \Sigma f = s$, then a classical result of Kato-Matsumoto [Kat73] tells us that $\widetilde{H}^k(F_{f,p}; \mathbb{Z}) \neq 0$ only for $n - s \leq k \leq n$, and the Milnor fiber is contractible if and only if $p$ is a non-singular point of $V(f)$. The majority of the study of non-isolated hypersurface singularities boils down to understanding these cohomology groups.

The next "easiest" case to examine after isolated singularities is, of course, hypersurfaces with one-dimensional singularities. Here, we know that $F_{f,p}$ can only have non-trivial cohomology in degrees $n$ and $n-1$, and still it is highly non-trivial to compute these groups in general. The general setting of Lê's conjecture is then interesting primarily because, for surfaces in $\mathbb{C}^3$, there is not "enough room" for complicated topological phenomena to happen, even with non-isolated singularities. We now state the precise conjecture:

**Question 1.1 (Lê's Conjecture [198]).** *Let $(V(f), \mathbf{0}) \subseteq (\mathbb{C}^3, 0)$ be a reduced complex analytic surface germ with $\dim_0 \Sigma f = 1$, such that the normalization $\pi : (\widetilde{V(f)}, 0) \to (V(f), 0)$ is* *smooth, and $\pi$ is a bijection. Then, $V(f)$ is isomorphic to the total space of an equisingular deformation of an irreducible plane curve singularity.*

Since the problem is local, we may assume $\widetilde{V(f)}$ is just $\mathbb{C}^2$. Additionally, we will not precisely define the general notion of "equisingular deformation" here, but it suffices to say that there exists a generic linear function $L : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ such that $V(L)$ transversely intersects $\Sigma f$ at 0, and for all $t \in \mathbb{C}$ small, the Milnor number $\mu_0(f_{|_{V(L-t)}})$ is independent of $t$. In this way, we can regard $V(f)$ as a family of plane curves $V(f, L-t)$ with isolated singularities that are all "the same".




Non-example: the normalization of the Whitney umbrella $y^2 = x^3 + zx^2$ is smooth, but not a bijection. Hence it is **not** an equisingular deformation of the cusp $y^2 = x^3$.

Despite this Conjecture having been around for nearly 40 years, it is only known to be true in a handful of special cases: when $\Sigma f$ contains a smooth curve, when $V(f)$ is a cyclic cover of a normal surface singularity, and when $f$ is a sum of two homogeneous forms, to name a few (see e.g., [Bob06],[Bob06] for a complete list of known cases). It is suspected that perhaps new theory must be developed to attack this problem, or that it may involve more of the interplay between the analytic and topological properties of surface germs.

Our approach to this problem is centered around the technical machinery of perverse sheaves and mixed Hodge modules. The bulk of my previous work (and my Ph.D. thesis!) concerned the study of non-isolated hypersurface singularities with smooth normalizations via perverse sheaves [Hep16],[Hep18],[Hep19a],[Hep19b]. In particular, I recover Bobadilla's result as a special case of the main results of [Hep18]. Via the language of perverse sheaves, it is then possible to rephrase the Conjecture in terms of the complex of vanishing cycles $\varphi_f[-1]\mathbb{Q}^\bullet_{\mathbb{C}^3}[3]$. We will discuss perverse sheaves in more detail in the following section, but in the meantime we loosely think of $\varphi_f[-1]\mathbb{Q}^\bullet_{\mathbb{C}^3}[3]$ as a complex of sheaves of finite dimensional $\mathbb{Q}$-vector spaces whose support is $V(f) \cap \Sigma f$, and for all $p \in \Sigma f$, there is a canonical isomorphism

$$\mathscr{H}^k(\varphi_f[-1]\mathbb{Q}^\bullet_{\mathbb{C}^3}[3])_p \simeq \widetilde{H}^{2+k}(F_{f,p}; \mathbb{Q})$$

There is a natural monodromy action $T_f$ on the cohomology of the Milnor fiber, given by allowing the values of $f$ to travel in a circle around the origin in $\mathbb{C}$. This extends to the level of the derived category to a natural isomorphism of perverse sheaves (also denoted $T_f$) on $\varphi_f[-1]\mathbb{Q}^\bullet_{\mathbb{C}^3}[3]$. The Milnor monodromy operator has a Jordan decomposition $T_f = T_f^u \circ T_f^s$ where $T_f^u$ is unipotent, and $T_f^s$ is semi-simple of finite order. For $\lambda \in \mathbb{Q}$, the (generalized) eigenspaces $\varphi_{f,\lambda} := \ker\{T_f^s - \lambda \cdot Id\}$ are perverse subsheaves of $\varphi_f[-1]\mathbb{Q}^\bullet_{\mathbb{C}^3}[3]$, and there is a natural splitting $\varphi_f[-1]\mathbb{Q}^\bullet_{\mathbb{C}^3}[3] \simeq \varphi_{f,1} \oplus \varphi_{f,\neq 1}$. Via my previous work in [Hep19b], the hypotheses of Lê's Conjecture imply the vanishing $\varphi_{f,1} = 0$; consequently, the non-unipotent vanishing cycles become the central object of study.

Perverse sheaves represent the topological perspective of our approach to proving this Conjecture; to see the analytic structure we have mentioned, we "enhance" these objects to mixed Hodge modules. These are, broadly speaking, perverse sheaves whose stalks are all mixed Hodge structures, and generically are variations of (polarizable) mixed Hodge structures. Together with Laurentiu Maxim at the University of Wisconsin-Madison (where I am currently is employed as a visiting assistant professor), we have reduced Lê's Conjecture to the following statement.

**Question 1.2** (H., Maxim)**.** *Let $V(f)$ be as in Problem 1.1. Then, the non-unipotent vanishing cycles $\varphi_{f,\neq 1}$ is a semi-simple mixed Hodge module that is pure of weight 2.*

Intuitively, semi-simplicity removes the "obstruction" to $V(f)$ being an equisingular deformation.

## 2. Irregular Microlocal Perversity and Subanalytic Geometry

My second project is broadly concerned with developing a microlocal criterion for irregular perversity, in the spirit of a corresponding result for classical perverse sheaves due to Kashiwara-Schapira (see Definition 10.3.7 and Theorem 10.3.12 of [Kas90]). This project has been the bulk of my postdoctoral work at UW-Madison with respect to the time required to learn such a cutting-edge framework, and in the sense of it being my first long-term research project largely proposed and planned on my own after graduate school.

The first main step in this project involves generalizing the notion of vanishing cycles to this setting as a method for probing microlocal behavior, in a way that both serves an analogous purpose for irregular perverse sheaves, as well as recovers the classical perverse sheaves when restricted to that subcategory [Hep21]. We will give some quick motivation for the classical microlocal perversity criterion, and afterward talk about generalized vanishing cycles and why we should "believe" a microlocal perversity criterion should exist for irregular perverse sheaves.

We will need to hit the ground running: our starting point for this topic will be with bounded, $\mathbb{C}$-constructible complex of sheaves of $\mathbb{Q}_X$-modules on a given complex manifold $X$, and denote the associated (derived) category of these objects by $D^b_{\mathbb{C}-c}(X)$. The notion of $\mathbb{C}$-constructibility here refers to the "sufficiently nice cohomology" statement I made in the introduction, where we say $\mathbf{F}^\bullet$ is $\mathbb{C}$-constructible if there is a locally finite collection of complex submanifolds $\{S_\alpha\}_\alpha$ of $X$ along which the cohomology sheaves $\mathscr{H}^k(\mathbf{F}^\bullet)_{|S_\alpha}$ are all locally constant and of finite rank. The category $\operatorname{Perv}(X)$ of perverse sheaves (with middle perversity, with coefficients in $\mathbb{Q}$) is then a full Abelian subcategory of this derived category, whose objects we can characterize **microlocally**, that is, by their local behavior on the cotangent bundle $T^*X \xrightarrow{\pi} X$.

Vanishing cycles allow one to probe the microlocal behavior of our $\mathbb{C}$-constructible complex $\mathbf{F}^\bullet$, in the sense that the cohomology sheaves of $\mathbf{F}^\bullet$ are locally constant "in the direction" of a covector $(p; d_pf) \in T^*X$ if and only if $\mathscr{H}^k(\varphi_f\mathbf{F}^\bullet)_p = 0$ for all $k \in \mathbb{Z}$. With this in mind, one calculates the **microsupport** $SS(\mathbf{F}^\bullet) \subset T^*X$, a conic Lagrangian subvariety of $T^*X$, as the complement of the set of all covectors $(p; \xi)$ for which $(\varphi_f\mathbf{F}^\bullet)_p \simeq 0$ for any choice of local holomorphic representative $f$ with $\xi = d_pf$.

With this in mind, we will now be able to state the result that is the conceptual foundation of this project. Although the full, precise statement of this theorem will be given, we will not go into most of the details or constructions that are mentioned (for sake of readability to non-specialists); instead, we will spend time talking about some of the underlying intuition behind this approach if one's goal is to understand perverse sheaves from a "geometric" perspective.

**Theorem 1** (**microlocal perversity criterion**, [Kas90] Theorem 10.3.12)**.** *A complex of sheaves $\mathbf{F}^\bullet \in D^b_{\mathbb{C}-c}(X)$ is a perverse sheaf on $X$ if and only if, for every non-singular point $(p;\xi)$ of $SS(\mathbf{F}^\bullet)$ such that the projection $\pi : SS(\mathbf{F}^\bullet) \to X$ has constant rank on a neighborhood of $(p;\xi)$, there exists a complex submanifold $Y \subset X$ and a complex of finite-dimensional vector spaces $M^\bullet \in D^b(Mod^f(\mathbb{Q}))$ such that $\mathbf{F}^\bullet \simeq M^\bullet_Y[\dim Y]$ in $D^b(X;(p;\xi))$ and $H^k(M^\bullet) = 0$ for $k \neq 0$.*

Intuitively, the microlocal perversity criterion says that if an object $\mathbf{F}^\bullet$ is locally constant along some stratum $S$ near a point $p \in S$, then it can only qualitatively change in behavior if one "leaves" the stratum along some covector in the conormal bundle to $S$ in $X$, and one has $SS(\mathbf{F}^\bullet) = T^*_S X$ in a small neighborhood over $p$ (in general, we would also need to consider limiting covectors from nearby, higher-dimensional strata). The change in behavior is computed using vanishing cycles, and the stalks $(\varphi_L[-1]\mathbf{F}^\bullet)_p$ are independent of the covectors $(p; d_pL)$ chosen generically in $T^*_S X$ near the fiber over $p$. We take a generic such stalk and set $M^\bullet := (\varphi_L[-1]\mathbf{F}^\bullet)_p$, which is a complex of finite dimensional $\mathbb{Q}$-vector spaces, and the corresponding complex of sheaves $M^\bullet_S[\dim S]$ is "microlocally isomorphic" to $\mathbf{F}^\bullet$ near $(p; d_pL)$. Consequently, the complex of sheaves $\mathbf{F}^\bullet$ is perverse if and only if this "change along $S$" object $M^\bullet$ is concentrated in cohomological degree 0, i.e., just a finite dimensional vector space. This perspective comes from the fundamental work of Goresky-MacPherson in stratified Morse theory [Gor88] (particularly, the notions of "splitting of local Morse data" and "normal Morse data" along a Whitney stratification), later generalized to microlocal Morse theory by Kashiwara-Schapira [Kas90]. **I have always found this to be the most intuitive and geometric perspective from which to view perverse sheaves**–

**Moral.** *Perverse sheaves are locally constant objects along strata, are as nice as possible when not locally constant, and all of this data is encoded microlocally by using vanishing cycles.*

It is then natural to conjecture the following:

**Question 2.1.** *Does a characterization analogous to the Microlocal Perversity Criterion (Theorem 1) exists for the category of irregular perverse sheaves* E-Perv$(\mathrm{I}\mathbb{Q}_X)$ *inside the ambient category of $\mathbb{C}$-constructible enhanced ind-sheaves* $\mathrm{E}^b_{\mathbb{C}-c}(\mathrm{I}\mathbb{Q}_X)$*?*

This is still a fairly new framework, and it may not be surprising that every characterization of the far more technical category irregular perverse sheaves has not been discovered; but, for many who work with classical perverse sheaves for topological applications, it is a important problem that holds back more widespread use of this machinery. One of the first steps toward this goal is to develop a suitable notion of "irregular vanishing cycles" that work in all dimensions, for any choice of (holomorphic) function $f : X \to \mathbb{C}$. Some progress has recently been made in this direction due to D'Agnolo-Kashiwara in 2020, where the authors give a construction of irregular vanishing cycles in the case where $X$ is one-dimensional

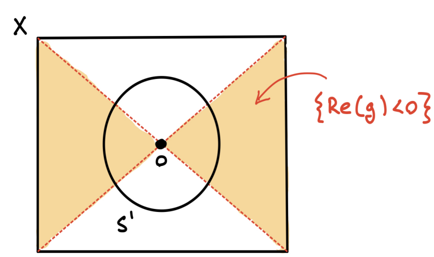
[D'A20] in terms of certain enhanced versions of the specialization and microlocalization functors (see Chapter 4 of [Kas90] for their classical setting, and [D'A21] for the enhanced setting).

Toward this goal for enhanced ind-sheaves, I have shown that there are two distinct–but Verdier-dual– functors that deserve the name of "irregular vanishing cycles" associated to a given holomorphic function $f : X \to \mathbb{C}$, which are denoted $\varphi_f^{+\infty}$ and $\varphi_f^{-\infty}$. These functors correspond to the two distinct ways in which an irregular local system on the complement of a given hypersurface $V(f)$ can be extended across that hypersurface. Irregular local systems are, intuitively, the fundamental building blocks of $\mathbb{C}$-constructible enhanced ind-sheaves, much like how local systems are the building blocks of classical $\mathbb{C}$-constructible sheaves in the derived category $D^b(X)$. This paper is currently in preparation, and will be on the arXiv early Fall 2021 [Hep21].

To visualize the two distinct extension phenomena mentioned above, it is easiest to work in the local dimension one setting, where $X$ is an open neighborhood of the origin in $\mathbb{C}$ and consider those objects having only one singularity at 0. Let $X^* := X \backslash \{0\}$. The prototypical irregular perverse sheaf in this setting is an **exponential enhanced ind-sheaf** (or just exponential object for short) associated to a meromorphic function $g$ on $X$ with pole at 0, denoted by $\mathbb{E}^{\mathrm{Re}(g)}_{X^*|X}[1]$. This exponential object corresponds (via the irregular Riemann-Hilbert correspondence) to the meromorphic connection $(\mathcal{O}_X(*0), d - dg)$ on $X$, whose flat sections are generated by the (meromorphic) exponential function $e^g$ on $X^*$. One should think of this exponential object as

$$\text{“the constant sheaf } \mathbb{Q}^\bullet_{X^*}[1]\text{”} \\ + \\ \text{“additional data at 0 determined by the exponent } g\text{”}$$

This additional data lives on the space of directions to 0 in $X$ ( i.e., on $S^1$), and keeps track of the asymptotic behavior of $\mathrm{Re}(g)$ as one approaches 0 along different angles from $X^*$ (we only care about the real part of $g$, since this is what governs the growth of $|e^{g(z)}| = e^{\mathrm{Re}(g(z))}$).




The regions of $X^*$ where $e^{\mathrm{Re}(g)}$ is bounded above, for the exponent $g(z) = \frac{-1}{z^2}$ (pictured in orange). The associated asymptotic data on $S^1$ is given by $\{|z| = \epsilon\} \cap \{\mathrm{Re}(g(z)) < 0\}$ for $0 < \epsilon \ll 1$.

Loosely speaking, if $z$ is a local holomorphic coordinate at 0, $\varphi_z^{+\infty}\mathbb{E}^{\mathrm{Re}(g)}_{X^*|X}[1]$ detects those angles $\theta \in S^1$ at which $\mathrm{Re}(g) \to +\infty$ when $z \to 0$ along a ray at angle $\theta$, and $\varphi_z^{-\infty}\mathbb{E}^{\mathrm{Re}(g)}_{X^*|X}[1]$ analogously detects those angles at which $\mathrm{Re}(g) \to -\infty$. Moreover, from this description, one immediately observes there is a duality

$$\varphi_z^{+\infty}\mathbb{E}^{\mathrm{Re}(g)}_{X^*|X}[1] \simeq \varphi_z^{-\infty}\mathbb{E}^{\mathrm{Re}(-g)}_{X^*|X}[1], \tag{1}$$

which implies a more general isomorphism for arbitrary irregular perverse sheaves with singularity at 0, and as well as the fact that $\varphi_z^{+\infty}\mathbf{K}^\bullet \neq 0$ if and only if $\varphi_z^{-\infty}\mathbf{K}^\bullet \neq 0$ when $\mathbf{K}^\bullet$ is an irregular perverse sheaf with singularity at 0 [Hep21]. In the exponential case, both vanishing cycles being zero implies the function $g$ must be bounded in a neighborhood of 0, and $\mathbb{E}^{\mathrm{Re}(g)}_{X^*|X}[1]$ extends to the irregular constant sheaf, denoted $\mathbb{Q}^{\mathrm{E}}_X[1]$.

Recalling the microlocal perversity criterion in dimension one, we find that a (classical) $\mathbb{C}$-constructible complex of sheaves $\mathbf{F}^\bullet \in D^b_{\mathbb{C}-c}(X)$ is perverse (say, with respect to the stratification $\{X^*, \{0\}\}$) if and only if the following conditions hold:

(a) Over any point $p \in X^*$, there is a microlocal isomorphism $\mathbf{F}^\bullet_{|X^*} \simeq M^\bullet_{X^*}[1]$ in $D^b(X;(p;0))$ for some finite-dimensional vector space $M$–it is not too hard to see that this vector space is obtained by taking the stalk of the cohomology sheaf $\mathscr{H}^{-1}(\mathbf{F}^\bullet)$ at the point $p$ (by $\mathbb{C}$-constructibility of $\mathbf{F}^\bullet$, $M$ is independent of the point $p \in X^*$ up to isomorphism).

(b) Over the origin $0 \in X$, there is a microlocal isomorphism $\mathbf{F}^\bullet \simeq \mathrm{R}\Gamma_{\{\mathrm{Re}(z)\geq 0\}}(\mathbf{F}^\bullet)$ in $D^b(X;(0;d_0 z))$, and it is a classic exercise that this "cohomology of $\mathbf{F}^\bullet$ with supports in $\{\mathrm{Re}(z) \geq 0\}$" object vanishes everywhere except at 0, where we have a canonical isomorphism of stalks

$$\mathrm{R}\Gamma_{\{\mathrm{Re}(z)\geq 0\}}(\mathbf{F}^\bullet)_{|0} \simeq \varphi_z[-1]\mathbf{F}^\bullet$$

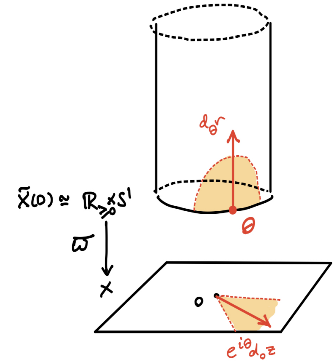
(cf, Exercise VIII.13, [Kas90]).

Now, in dimension one, **we should "believe" that an "irregular" perversity condition might hold** for a general object $\mathbf{K}^\bullet \in \mathrm{E}^b_{\mathbb{C}-c}(\mathrm{I}\mathbb{Q}_X)$ for the following reason. We can "forget" the extra enhanced data of $\mathbf{K}^\bullet$ to obtain a classical sheaf $\mathrm{sh}_X(\mathbf{K}^\bullet)$ (called the **associated sheaf of $\mathbf{K}^\bullet$**); this associated sheaf will belong to the category $D^b_{\mathbb{C}-c}(X)$, and moreover $\mathrm{sh}_X(\mathbf{K}^\bullet)$ is a bonafide classical perverse sheaf whenever the enhanced ind-sheaf $\mathbf{K}^\bullet$ is an irregular perverse sheaf on $X$ (see Lemma 3.5 of [D'A20]). Consequently, the associated sheaf $\mathrm{sh}_X(\mathbf{K}^\bullet)$ must satisfy the two microlocal perversity conditions $(a)$ and $(b)$ mentioned above, as well as an additional condition $(c)$ concerning the asymptotic data of $\mathbf{K}^\bullet$ living on $S^1$. Intuitively, condition $(c)$ is obtained from condition $(b)$ as one "rotates" the covector $d_0 z$ used for the microlocal isomorphism in $D^b(X;(0;d_0 z)$ to different angles $e^{i\theta}d_0 z \in S^1 \simeq (T^*_0 X - \{(0;0)\})/\mathbb{R}^+$.

Before stating this final condition, we need the notion of the **real oriented blow-up of $X$ at** 0,

$$\varpi : \widetilde{X}(0) \simeq \mathbb{R}_{\geq 0} \times S^1 \longrightarrow X$$
$$(r,\theta) \mapsto z = re^{i\theta}$$

That is, $\widetilde{X}(0)$ is just the space of polar coordinates on $\mathbb{C}$ centered at 0, in some small neighborhood of the origin.




A sectorial neighborhood of an angle $\theta \in S^1$, pictured in $X$ and in $\widetilde{X}(0)$. The covector $e^{i\theta} d_0 z$ lifts to the inward-pointing conormal covector $d_\theta r$ to $S^1 \simeq \partial \widetilde{X}(0)$ in $\widetilde{X}(0)$.

**Theorem 2** (**condition (c)**, [Hep21]). *Let* $\mathbf{K}^\bullet \in \mathrm{E}^b_{\mathbb{C}-c}(\mathrm{I}\mathbb{Q}_X)$. *Then,* $\mathbf{K}^\bullet \in \text{E-Perv}(\mathrm{I}\mathbb{Q}_X)$ *if and only if,* $\mathrm{sh}_X(\mathbf{K}^\bullet)$ *satisfied conditions (a) and (b) above, and for any angle* $\theta$*, there is a complex of finite-dimensional vector spaces* $L^\bullet \in D^b(\mathrm{Mod}^f(\mathbb{Q}))$ *and an isomorphism*

$$\mathrm{sh}_{\widetilde{X}(0)}(\mathrm{E}\varpi^{-1}\mathbf{K}^\bullet) \simeq L^\bullet_{S^1} \text{ in } D^b(\widetilde{X}(0); d_\theta r) \tag{2}$$

*and* $H^k(L^\bullet) = 0$ *for* $k \neq 0$.

Note that $\widetilde{X}(0)$ is a smooth manifold with boundary $\partial \widetilde{X}(0) \simeq S^1$, and $d_\theta r$ corresponds to (the unit-lenght covector dual to) the inward-pointing normal vector to $S^1$ in $\widetilde{X}(0)$ at $\theta$.

Making this collection of microlocal isomorphism conditions precise for higher dimensions one of my long-term goals for this project.

In the recent literature, the functors $\varphi_f^{\pm\infty}$ for enhanced ind-sheaves described in [Hep21] are equivalent (via Riemann-Hilbert) to two recent constructions for **holonomic $\mathscr{D}$-modules** by Claude Sabbah in early 2021 [Sab21]. In *loc. cit.*, Sabbah considers various functors on the (derived) category of holonomic $\mathscr{D}$-modules which extend the classical construction of the nearby cycles functor $\psi_f$ for $\mathscr{D}$-modules, for the purpose of extracting information about local sections of a given $\mathscr{D}$-module $\mathcal{M}$ with different growth conditions along $V(f)$ (such as having moderate growth, rapid decay, or greater than moderate growth along $V(f)$, denoted $\psi_f^{mod}$, $\psi_f^{rd}$, and $\psi_f^{>mod}$, resp.). More precisely, $\varphi_f^{+\infty}$ is equivalent to $\psi_f^{>mod}$, and $\varphi_f^{-\infty}$ is equivalent to $\psi_f^{rd}$. Additionally, the functors $\varphi_f^{\pm\infty}$ agree (up to a shift that preserves perversity) with the functors $\lambda^{rb}_{V(f)}$ and $\widetilde{\lambda}^{rb}_{V(f)}$ (resp.) described by D'Agnolo-Kashiwara's in the case where $V(f)$ is non-singular (cf, Appendix A of [D'A20]), although we note that these functors are not themselves described as "enhanced vanishing cycles", but are instead provided as an alternative characterization of their enhanced vanishing cycles functor when $X$ is one-dimensional (given by an enhancement of the classical microlocalization functor).

However, the flexibility of this framework is that one can now handle arbitrary irregular local systems in arbitrary dimensions, i.e., general exponential objects $\mathbb{E}^h_{U|X}$, where $U \subset X$ is an open subanalytic subset and $h : U \to \mathbb{R}$ is **any** subanalytic continuous function. As a consequence of this, I hope in the future to be able to apply these new functors to study the singularities of real subanalytic spaces, and the Lipschitz geometry of complex analytic spaces (both are areas that are famously difficult to study sheaf-theoretically). In particular, these vanishing cycle functors appear to be suited to computing a dual, cohomological version of the new moderately discontinuous homology theory of Bobadilla-Heinze-Pe Pereira-Sampaio [Fer19] to study germs of metric subanalytic spaces.

## References


[198] "Nœuds, tresses et singularités. Conference." In: 1983.

[Bei81] Beilinson, A. A., Bernstein, J., Deligne, P. *Faisceaux pervers*. Vol. 100. Astérisque. Soc. Math. France, 1981.

[Bob06] Bobadilla, J. "A Reformulation of Lê's Conjecture". In: *Indag. Math.* 17, no. 3 (2006), pp. 345–352.

[D'A16] D'Agnolo, A. and Kashiwara, M. "Riemann-Hilbert correspondence for holonomic D-modules". In: *Publ. Math. Inst. Hautes Études Sci.* 123 (2016), pp. 69–197.

[D'A20] D'Agnolo, A. and Kashiwara, M. "Enhanced nearby and vanishing cycles in dimension one and Fourier transform". In: (2020). arXiv: 2002.11341 [math.AG].

[D'A21] D'Agnolo, A. and Kashiwara, M. "Enhanced specialization and microlocalization". In: *Selecta Math.* 27 (2021).

[Fer19] Fernandez de Bobadilla, J., Heinze, S., Pe Pereira, M., and Sampaio, E. "Moderately discontinuous homology". In: *ArXiv e-prints* (2019). arXiv: 1910.12552 [math.AG].

[Gor88] Goresky, M. and MacPherson, R. *Stratified Morse Theory*. Vol. 14. Ergeb. der Math. Springer-Verlag, 1988.

[Hep16] Hepler, B. and Massey, D. "Perverse Results on Milnor Fibers inside Parameterized Hypersurfaces". In: Publ. RIMS Kyoto Univ. 52 (2016), pp. 413–433.

[Hep18] Hepler, B. "Deformation Formulas for Parameterizable Hypersurfaces". In: *ArXiv e-prints (accepted, Ann. Inst. Fourier)* (2018). arXiv: 1711.11134 [math.AG].

[Hep19a] Hepler, B. "Rational Homology Manifolds and Hypersurface Normalizations". In: *Proc. Amer. Math. Soc.* 147 (2019), pp. 1605–1613.

[Hep19b] Hepler, B. "The Weight Filtration on the Constant Sheaf on a Parameterized Space". In: (2019). arXiv: 1811.04328 [math.AG].

[Hep21] Hepler, B. "Vanishing Cycles for Irregular Local Systems". In: *in prep.* (2021).

[Kas90] Kashiwara, M. and Schapira, P. *Sheaves on Manifolds*. Vol. 292. Grund. math. Wissen. Springer-Verlag, 1990.

[Kat73] Kato, M. and Matsumoto, Y. "On the connectivity of the Milnor fibre of a holomorphic function at a critical point". In: *Proc. of 1973 Tokyo manifolds conf.* (1973), pp. 131–136.

[Mil68] Milnor, J. *Singular Points of Complex Hypersurfaces*. Vol. 77. Annals of Math. Studies. Princeton Univ. Press, 1968.

[Sab21] Sabbah, C. "Moderate and Rapid-Decay Nearby Cycles for Holonomic D-Modules". In: *preprint* (2021).