AMS-SIMONS TRAVEL GRANT RESEARCH SUMMARY

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Non-Specialist Summary

The author's research is centered around a nearly 40-year-old open problem by Lê Dũng Tráng concerning the equisingularity of germs of complex analytic surfaces $V(f) := \{f = 0\}$ in $\mathbb{C}^3$ with one-dimensional singular locus. Lê's Conjecture concerns the deep relationship between the topological and analytic properties of surface germs: if the link of $V(f)$ (the intersection of $V(f)$ with a sufficiently small sphere centered at 0) is homeomorphic to a 3-sphere, then the singular locus of $V(f)$ is a smooth curve at 0. The link of a surface is one of the most important pieces of data associated to a singularity, and this hypothesis places strict constraints on the local topology of $V(f)$ at 0–in particular, it implies that the normalization of $V(f)$ is smooth, and is a bijection. One pictures $V(f)$ as "folding up" $\mathbb{C}^2$, with the "creases" corresponding to the singular locus. The author's approach, together with Laurentiu Maxim, is via the machinery of Saito's mixed Hodge modules. We give some motivation for this Conjecture, as well as an equivalent formulation in the language of mixed Hodge modules below in the research summary.

1. Research Summary: Lê's Conjecture

This project is the investigation of a classic conjecture (due to Lê [198]) in the field of singularities of complex analytic spaces, specifically on the so-called "equisingularity" of certain surfaces with non-isolated singularities inside $\mathbb{C}^3$. Let us briefly recall some of the essential notions from singularity theory.

The study of (the topology of) complex hypersurfaces with **isolated** singularities largely started with the foundational work of Milnor [Mil68]; the local, ambient topological type of a hypersurface $V(f) \subseteq \mathbb{C}^{n+1}$ at a singular point $p \in V(f)$ is completely determined by a fibration (called the Milnor fibration) defined on a "tube" around the hypersurface near p. More precisely, for $0 < \delta \ll \epsilon \ll 1$, the defining function f restricts to a smooth, locally trivial fibration

$$\hat{f} : B_\epsilon(p) \cap f^{-1}(\partial \mathbb{D}_\delta) \to \partial \mathbb{D}_\delta$$

where $B_\epsilon(p)$ is an open ball of radius ϵ at p in $\mathbb{C}^{n+1}$ (with respect to any Riemannian metric), and $\mathbb{D}_\delta$ is a disk of radius δ around 0 in $\mathbb{C}$. The fiber of $\hat{f}$ is called the **Milnor fiber** of f at p, denoted $F_{f,p}$, is a compact, orientable manifold of dimension $2n$ that is homotopy equivalent to a finite bouquet of n-spheres. The number of such spheres is called the **Milnor number** of f at p, and is denoted $\mu_p(f)$.

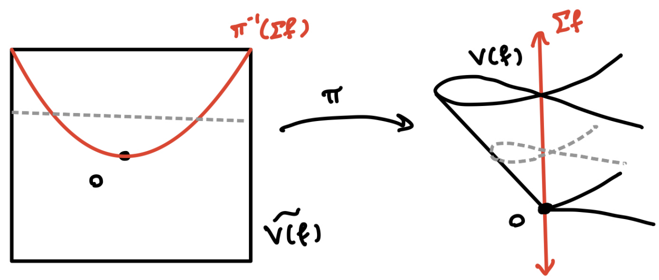
Milnor's fibration still exists in the more general context of hypersurfaces with non-isolated singularities, but the associated Milnor fiber is no longer as nice as in the isolated case. It still has the homotopy-type of a finite CW complex, and if the singular locus Σf has $\dim_0 \Sigma f = s$, then a classical result of Kato-Matsumoto [Kat73] tells us that $\widetilde{H}^k(F_{f,p}; \mathbb{Z}) \neq 0$

only for $n - s \leq k \leq n$, and the Milnor fiber is contractible if and only if p is a non-singular point of $V(f)$. The majority of the study of non-isolated hypersurface singularities boils down to understanding these cohomology groups.

The next "easiest" case to examine after isolated singularities is, of course, hypersurfaces with one-dimensional singularities. Here, we know that $F_{f,p}$ can only have non-trivial cohomology in degrees n and $n-1$, and still it is highly non-trivial to compute these groups in general. The general setting of Lê's conjecture is then interesting primarily because, for surfaces in $\mathbb{C}^3$, there is not "enough room" for complicated topological phenomena to happen, even with non-isolated singularities. We now state the precise conjecture:

Problem 1.1 (Lê's Conjecture [198])**.** *Let $(V(f), \mathbf{0}) \subseteq (\mathbb{C}^3, 0)$ be a reduced complex analytic surface with $\dim_0 \Sigma f = 1$, such that the normalization $\pi : (\widetilde{V(f)}, 0) \to (V(f), 0)$ is smooth, and π is a bijection. Then, $V(f)$ is isomorphic to the total space of an equisingular deformation of an irreducible plane curve singularity.*

Since the problem is local, we may assume $\widetilde{V(f)}$ is just $\mathbb{C}^2$. Additionally, we will not precisely define the general notion of "equisingular deformation" here, but it suffices to say that there exists a generic linear function $L : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ such that $V(L)$ transversely intersects Σf at 0, and for all $t \in \mathbb{C}$ small, the Milnor number $\mu_0(f_{|_{V(L-t)}})$ is independent of t. In this way, we can regard $V(f)$ as a family of plane curves $V(f, L-t)$ with isolated singularities that are all "the same".



Non-example: the normalization of the Whitney umbrella $y^2 = x^3 + zx^2$ is smooth, but not a bijection.
Hence it is **not** an equisingular deformation of the cusp $y^2 = x^3$.

Despite this Conjecture having been around for nearly 40 years, it is only known to be true in a handful of special cases: when Σf contains a smooth curve, when $V(f)$ is a cyclic cover of a normal surface singularity, and when f is a sum of two homogeneous forms, to name a few (see e.g., [Fer06],[Fer06] for a complete list of known cases). It is suspected that perhaps new theory must be developed to attack this problem, or that it may involve more of the interplay between the analytic and topological properties of surface germs.

The author's approach to this problem is centered around the technical machinery of perverse sheaves and mixed Hodge modules. The bulk of the author's previous work (and Ph.D. thesis!) concerned the study of non-isolated hypersurface singularities with smooth normalizations via perverse sheaves [Hep16],[Hep17],[Hep19a],[Hep19b]. In particular, we recover Bobadilla's result as a special case of the main results of [Hep17]. Via the language of perverse sheaves, it is then possible to rephrase the Conjecture in terms of the complex of vanishing cycles $\varphi_f[-1]\mathbb{Q}^\bullet_{\mathbb{C}^3}[3]$. We will not go into the details of the language of perverse sheaves, or the derived category here for the sake of brevity, but loosely $\varphi_f[-1]\mathbb{Q}^\bullet_{\mathbb{C}^3}[3]$ is a complex of sheaves of finite dimensional $\mathbb{Q}$-vector spaces whose support is $V(f) \cap \Sigma f$, and for all $p \in \Sigma f$, there is a canonical isomorphism

$$H^k(\varphi_f[-1]\mathbb{Q}^\bullet_{\mathbb{C}^3}[3])_p \cong \widetilde{H}^{2+k}(F_{f,p};\mathbb{Q})$$

There is a natural monodromy action T_f on the cohomology of the Milnor fiber, given by allowing the values of f to travel in a circle around the origin in $\mathbb{C}$. This extends to the level of the derived category to a natural isomorphism of perverse sheaves (also denoted T_f) on $\varphi_f[-1]\mathbb{Q}^\bullet_{\mathbb{C}^3}[3]$. The Milnor monodromy operator has a Jordan decomposition $T_f = T_f^u \circ T_f^s$ where T_f^u is unipotent, and T_f^s is semi-simple of finite order. For $\lambda \in \mathbb{Q}$, the (generalized) eigenspaces $\varphi_{f,\lambda} := \ker\{T_f^s - \lambda \cdot Id\}$ are perverse subsheaves of $\varphi_f[-1]\mathbb{Q}^\bullet_{\mathbb{C}^3}[3]$, and there is a natural splitting $\varphi_f[-1]\mathbb{Q}^\bullet_{\mathbb{C}^3}[3] \cong \varphi_{f,1} \oplus \varphi_{f,\neq 1}$. Via the author's work in [Hep19b], the hypotheses of Lê's Conjecture imply the vanishing $\varphi_{f,1} = 0$; consequently, the non-unipotent vanishing cycles become the central object of study.

Perverse sheaves represent the topological perspective of the author's approach to proving this Conjecture; to see the analytic structure we have mentioned, we "enhance" these objects to mixed Hodge modules. These are, broadly speaking, perverse sheaves whose stalks are all mixed Hodge structures, and generically are variations of (polarizable) mixed Hodge structures. Together with Laurentiu Maxim at the University of Wisconsin-Madison (where the author currently is employed as a postdoc), we have reduced Lê's Conjecture to the following statement.

Problem 1.2 (H., Maxim)**.** *Let $V(f)$ be as in Problem 1.1. Then, the non-unipotent vanishing cycles $\varphi_{f,\neq 1}$ is a semi-simple mixed Hodge module that is pure of weight 2.*

Intuitively, semi-simplicity removes the "obstruction" to $V(f)$ being an equisingular deformation.

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