The Weight Filtration for Parameterized Surfaces

Brian Hepler
University of Wisconsin-Madison

February 10th, 2020
The Setting

- The central object of study for this work is a perverse sheaf called the \textbf{comparison complex}, first defined and explored by H. and David Massey in 2016 [6] (where we originally referred to it as the \textbf{multiple-point complex}), and in several subsequent papers by H. [2017,[3]], [2018,[4]], [2019,[5]] and Massey [2018,[7]].
The Setting

• The central object of study for this work is a perverse sheaf called the \textbf{comparison complex}, first defined and explored by H. and David Massey in 2016 [6] (where we originally referred to it as the \textbf{multiple-point complex}), and in several subsequent papers by H. [2017,[3],[2018,[4],[2019,[5]]] and Massey [2018,[7]].

• This perverse sheaf, denoted $N^\bullet_X$, is defined on any pure-dimensional (locally reduced) complex analytic space $X$ for which the constant sheaf $\mathbb{Q}^\bullet_X[\dim X]$ is perverse.
The central object of study for this work is a perverse sheaf called the **comparison complex**, first defined and explored by H. and David Massey in 2016 [6] (where we originally referred to it as the **multiple-point complex**), and in several subsequent papers by H. [2017,[3],[2018,[4]],][2019,[5]] and Massey [2018,[7]].

This perverse sheaf, denoted $\mathbf{N}_{X}^{\bullet}$, is defined on any pure-dimensional (locally reduced) complex analytic space $X$ for which the constant sheaf $\mathbb{Q}_{X}^{\bullet}[\dim X]$ is perverse.

$\mathbf{N}_{X}^{\bullet}$ “compares" the two fundamental perverse sheaves on such a space, the constant sheaf $\mathbb{Q}_{X}^{\bullet}[n]$ and the intersection cohomology complex $\mathbf{IC}_{X}^{\bullet}$ with constant coefficients, in a way that detects (some of) the singularities of $X$. 
Fundamental Short Exact Sequence

- There is a canonical surjection of perverse sheaves
  \( \mathbb{Q}^\bullet_X[n] \rightarrow \mathbf{IC}_X \rightarrow 0 \) (over \( \mathbb{Q} \) or \( \mathbb{Z} \));
Fundamental Short Exact Sequence

• There is a canonical surjection of perverse sheaves
  \( \mathbb{Q}^\bullet_X [n] \to \mathbf{IC}^\bullet_X \to 0 \) (over \( \mathbb{Q} \) or \( \mathbb{Z} \)); since the category of perverse sheaves on \( X \) is Abelian, we obtain a short exact sequence

  \[
  0 \to N^\bullet_X \to \mathbb{Q}^\bullet_X [n] \to \mathbf{IC}^\bullet_X \to 0.
  \]

  We will refer to this as the \textbf{fundamental short exact sequence} of perverse sheaves on \( X \).
Fundamental Short Exact Sequence

- There is a canonical surjection of perverse sheaves $\mathbb{Q}_X^\bullet[n] \to \mathbf{IC}_X^\bullet \to 0$ (over $\mathbb{Q}$ or $\mathbb{Z}$); since the category of perverse sheaves on $X$ is Abelian, we obtain a short exact sequence

$$0 \to N_X^\bullet \to \mathbb{Q}_X^\bullet[n] \to \mathbf{IC}_X^\bullet \to 0.$$  

We will refer to this as the fundamental short exact sequence of perverse sheaves on $X$.

- From this SES, we see that the support of $N_X^\bullet$ is contained in the singular locus of $X$, and the stalk cohomology of $N_X^\bullet$ is

$$H^i(N_X^\bullet)_p = \tilde{H}^{i-1}(K_{X,p}; \mathbf{IC}_X^\bullet),$$

where $\tilde{H}$ denotes reduced hypercohomology, and $K_{X,p} = X \cap S_\epsilon(p)$ denotes the real link of $X$ at $p$, for $0 < \epsilon \ll 1$. 
Parameterized Spaces, I

- Recall that a (reduced) complex analytic space $Y$ is a **Rational Homology Manifold** (or, $\mathbb{Q}$-homology manifold) if the natural morphism $\mathbb{Q}^\bullet_Y[n] \to \mathbf{IC}^\bullet_Y$ is an isomorphism in the derived category $D^b_c(Y)$ [Borho-MacPherson 1982 [1]].
Parameterized Spaces, I

- Recall that a (reduced) complex analytic space \( Y \) is a **Rational Homology Manifold** (or, \( \mathbb{Q} \)-homology manifold) if the natural morphism \( \mathbb{Q}^\bullet_Y[n] \rightarrow \mathcal{IC}^\bullet_Y \) is an isomorphism in the derived category \( D^b_c(Y) \) [Borho-MacPherson 1982 [1]]. That is, if \( N^\bullet_Y = 0 \).
Parameterized Spaces, I

- Recall that a (reduced) complex analytic space $Y$ is a **Rational Homology Manifold** (or, $\mathbb{Q}$-homology manifold) if the natural morphism $\mathbb{Q}^\bullet_Y[n] \to \mathbf{IC}^\bullet_Y$ is an isomorphism in the derived category $D^b_c(Y)$ [Borho-MacPherson 1982 [1]]. That is, if $N^\bullet_Y = 0$.

- Let $\pi: \tilde{X} \to X$ be the normalization of $X$. We say $X$ is a **parameterized space** if its normalization $\tilde{X}$ is a $\mathbb{Q}$-homology manifold. Such spaces are characterized by the following theorem.
Parameterized Spaces, I

- Recall that a (reduced) complex analytic space $Y$ is a **Rational Homology Manifold** (or, $\mathbb{Q}$-homology manifold) if the natural morphism $\mathbb{Q}^\bullet_Y[n] \to \mathbf{IC}^\bullet_Y$ is an isomorphism in the derived category $D^b_c(Y)$ [Borho-MacPherson 1982 [1]]. That is, if $N^\bullet_Y = 0$.

- Let $\pi : \tilde{X} \to X$ be the normalization of $X$. We say $X$ is a **parameterized space** if its normalization $\tilde{X}$ is a $\mathbb{Q}$-homology manifold. Such spaces are characterized by the following theorem.

**Theorem (H. 2018,[4])**

$X$ is a parameterized space if and only if the stalk cohomology of $N^\bullet_X$ is concentrated in degree $-n + 1$. 
Parameterized Spaces, I

• Recall that a (reduced) complex analytic space $Y$ is a **Rational Homology Manifold** (or, $\mathbb{Q}$-homology manifold) if the natural morphism $\mathbb{Q}^\bullet_Y[n] \to \text{IC}^\bullet_Y$ is an isomorphism in the derived category $D^b_c(Y)$ [Borho-MacPherson 1982 [1]]. That is, if $N^\bullet_Y = 0$.

• Let $\pi : \tilde{X} \to X$ be the normalization of $X$. We say $X$ is a **parameterized space** if its normalization $\tilde{X}$ is a $\mathbb{Q}$-homology manifold. Such spaces are characterized by the following theorem.

**Theorem (H. 2018,[4])**

$X$ is a parameterized space if and only if the stalk cohomology of $N^\bullet_X$ is concentrated in degree $-n + 1$.

• When $X$ is parameterized, the fundamental SES is then

\[ 0 \to N^\bullet_X \to \mathbb{Q}_X^\bullet[n] \to \pi_* \mathbb{Q}_{\tilde{X}}^\bullet[n] \to 0, \]

(since $\pi$ is a small map) and the canonical map $\mathbb{Q}_X^\bullet[n] \to \pi_* \mathbb{Q}_{\tilde{X}}^\bullet[n]$ coincides with the diagonal embedding in stalk cohomology.
Parameterized Spaces, II

When $X$ is parameterized, $N^\bullet_X$ has the following properties
When $X$ is parameterized, $N^\bullet_X$ has the following properties

- The stalk cohomology of $N^\bullet_X$ in degree $-n + 1$ is very easy to compute: for all $p \in X$,

$$H^{-n+1}(N^\bullet_X)_p \cong \mathbb{Q} |\pi^{-1}(p)|^{-1}.$$
When $X$ is parameterized, $\mathbf{N}_X^\bullet$ has the following properties

- The stalk cohomology of $\mathbf{N}_X^\bullet$ in degree $-n+1$ is very easy to compute: for all $p \in X$,

$$H^{-n+1}(\mathbf{N}_X^\bullet)_p \cong \mathbb{Q}|\pi^{-1}(p)|^{-1}.$$ 

- The support $D_X := \text{supp} \mathbf{N}_X^\bullet$ is purely $(n-1)$-dimensional, so $D_X \neq \emptyset$ implies $\Sigma X$ is purely $(n-1)$-dimensional. Moreover, the above stalk cohomology description of $\mathbf{N}_X^\bullet$ implies

$$D_X = \{ p \in X \mid |\pi^{-1}(p)| > 1 \}.$$ 

- We assume throughout that $D_X = \Sigma X$. 


Why Should You Care?

Our original purpose for studying $N^\bullet_X$ arose from understanding the\ncomplex link of parameterized spaces at a given point:
Our original purpose for studying $N^\bullet_X$ arose from understanding the **complex link** of parameterized spaces at a given point:

- Given a finite, generically one-to-one map $(\mathbb{C}^n, S) \xrightarrow{\pi} (\mathbb{C}^{n+1}, 0)$, $S$ a finite subset of $\mathbb{C}^n$. Then, $\text{im} \pi = V(f)$ for some analytic function $f$ on $\mathbb{C}^{n+1}$, and $\pi$ is the normalization of $V(f)$.

- Then, given a generic linear form $L$ on $\mathbb{C}^{n+1}$, the Milnor number of $L|_{V(f)}$ at 0 is given by the formula
Why Should You Care?

Our original purpose for studying $N^\bullet_X$ arose from understanding the **complex link** of parameterized spaces at a given point:

- Given a finite, generically one-to-one map $(\mathbb{C}^n, S) \xrightarrow{\pi} (\mathbb{C}^{n+1}, 0)$, $S$ a finite subset of $\mathbb{C}^n$. Then, $\text{im}\pi = V(f)$ for some analytic function $f$ on $\mathbb{C}^{n+1}$, and $\pi$ is the normalization of $V(f)$.

- Then, given a generic linear form $L$ on $\mathbb{C}^{n+1}$, the Milnor number of $L|_{V(f)}$ at $0$ is given by the formula

$$
\mu_0 \left( L|_{V(f)} \right) = \sum_{p \in S} \mu_p (L \circ \pi) \\
+ (-1)^{n-1} \left[ -1 + |\pi^{-1}(0)| + \sum_{k \geq 2} (k - 1) \chi(\mathbb{L}_{V(f),0} \cap V(f)_k) \right]
$$

where $\mathbb{L}_{V(f),0}$ is the complex link of $V(f)$ at $0$, and $V(f)_k = \{ p \in V(f) | |\pi^{-1}(p)| = k \}$. 
In Mather's "Nice Dimensions" ($\mu < 15$), $\mu_0(L|\nu(f),2)$ agrees with the image Milnor number of the map $\pi|\pi - 1(V(L),\nu(f))$, when $\pi$ is a stabilization of $\pi|\pi - 1(V(L))$. 

![Diagram of a mathematical concept involving maps and dimensions with arrows and labels.](image)
In Mather’s “Nice Dimensions” \((n < 15)\), \(\mu_0 \left( L_{|V(f)} \right)\) agrees with the **image Milnor number** of the map \(\pi|_{\pi^{-1}(V(L))}\), when \(\pi\) is a stabilization of \(\pi|_{\pi^{-1}(V(L))}\).
• When the stabilization $\pi$ of $\pi|_{\pi^{-1}(V(L))}$ is sufficiently generic (so that the origin in $V(f, L)$ splits into $\delta$ double points), we recover Milnor’s double point formula

$$\mu_0(f|_{V(L)}) = 2\delta - r + 1$$

where $r$ is the number of irreducible components of $V(f, L)$ at $0$.

• When $\pi$ is not a generic deformation, we obtain the formula

$$\mu_0(f|_{V(L)}) = -m(0) + \sum_{p \in B_\epsilon \cap V(L - \xi)} \left( m(p) + \mu_p(f|_{V(L - \xi)}) \right),$$

where $m(p) = |\pi^{-1}(p)| - 1 = \dim_{\mathbb{Q}} H^{-1}(N^\bullet_{V(f)})_p$.

• This formula can be use to prove Lê’s Conjecture in the case where $\pi$ is a corank one map.
Question: When is $N_X$ Semi-Simple?
Question: When is $\mathbf{N}_X^\bullet$ Semi-Simple?

- Over $\mathbb{Q}$, the category of perverse sheaves on $X$ is locally Artinian, with simple objects corresponding to intersection cohomology complexes $\mathbf{IC}_Z^\bullet(\mathcal{L})$ supported on irreducible subvarieties $Z$ of $X$ and coefficients in irreducible local systems $\mathcal{L}$. 
Question: When is $N_X^\bullet$ Semi-Simple?

- Over $\mathbb{Q}$, the category of perverse sheaves on $X$ is locally Artinian, with simple objects corresponding to intersection cohomology complexes $IC^\bullet_Z(\mathcal{L})$ supported on irreducible subvarieties $Z$ of $X$ and coefficients in irreducible local systems $\mathcal{L}$.

- From fundamental short exact sequence
  
  $$0 \to N_X^\bullet \to \mathbb{Q}_X^\bullet[n] \to IC_X^\bullet \to 0,$$

  we note that $IC_X^\bullet$ is a semi-simple perverse sheaf. It is then a natural question to ask:
Question:
When is $N^\bullet_X$ Semi-Simple?

- Over $\mathbb{Q}$, the category of perverse sheaves on $X$ is locally Artinian, with \textbf{simple objects} corresponding to intersection cohomology complexes $IC^\bullet_Z(\mathcal{L})$ supported on irreducible subvarieties $Z$ of $X$ and coefficients in irreducible local systems $\mathcal{L}$.

- From fundamental short exact sequence

$$0 \to N^\bullet_X \to \mathbb{Q}^\bullet_X[n] \to IC^\bullet_X \to 0,$$

we note that $IC^\bullet_X$ is a semi-simple perverse sheaf. It is then a natural question to ask:

\textbf{Question}

Is $N^\bullet_X$ ever semi-simple, so that $\mathbb{Q}^\bullet_X[n]$ is an extension of semi-simples?
Hodge Theory Strategy: The Weight Filtration on $\mathbb{Q}^\bullet_X[n]$

- Via Morihiko Saito's theory of mixed Hodge modules [Saito 1990, [8]], the natural quotient morphism $\mathbb{Q}^\bullet_X[n] \to \mathbf{IC}_X$ induces an isomorphism

$$\text{Gr}^W_n \mathbb{Q}^\bullet_X[n] \cong \mathbf{IC}_X,$$

where $\text{Gr}^W_n \mathbb{Q}^\bullet_X[n]$ is the $n$-th graded piece of the weight filtration on $\mathbb{Q}^\bullet_X[n]$, considered as a mixed Hodge module on $X$. 
Hodge Theory Strategy: The Weight Filtration on $\mathbb{Q}^\bullet_X[n]$

- Via Morihiko Saito’s theory of mixed Hodge modules [Saito 1990, [8]], the natural quotient morphism $\mathbb{Q}^\bullet_X[n] \to \mathbf{IC}_X$ induces an isomorphism

$$\text{Gr}_n^W \mathbb{Q}^\bullet_X[n] \cong \mathbf{IC}_X,$$

where $\text{Gr}_n^W \mathbb{Q}^\bullet_X[n]$ is the $n$-th graded piece of the weight filtration on $\mathbb{Q}^\bullet_X[n]$, considered as a mixed Hodge module on $X$.

- Consequently, the fundamental short exact sequence identifies the comparison complex $\mathbf{N}^\bullet_X$ with $W_{n-1} \mathbb{Q}^\bullet_X[n]$, and $\mathbf{N}^\bullet_X$ inherits the structure of a mixed Hodge module with weight filtration

$$W_k \mathbf{N}^\bullet_X = W_k \mathbb{Q}^\bullet_X[n] \quad \text{for } k \leq n - 1.$$
Hodge Theory Strategy:
The Weight Filtration on $\mathbb{Q}^\bullet_X[n]$

- Via Morihiko Saito’s theory of mixed Hodge modules [Saito 1990, [8]], the natural quotient morphism $\mathbb{Q}^\bullet_X[n] \to \mathbf{IC}^\bullet_X$ induces an isomorphism
  \[ \text{Gr}_n^W \mathbb{Q}^\bullet_X[n] \cong \mathbf{IC}^\bullet_X, \]
  where $\text{Gr}_n^W \mathbb{Q}^\bullet_X[n]$ is the $n$-th graded piece of the weight filtration on $\mathbb{Q}^\bullet_X[n]$, considered as a mixed Hodge module on $X$.

- Consequently, the fundamental short exact sequence identifies the comparison complex $\mathbf{N}^\bullet_X$ with $W_{n-1} \mathbb{Q}^\bullet_X[n]$, and $\mathbf{N}^\bullet_X$ inherits the structure of a mixed Hodge module with weight filtration
  \[ W_k \mathbf{N}^\bullet_X = W_k \mathbb{Q}^\bullet_X[n] \quad \text{for } k \leq n - 1. \]

- Answering our question is then equivalent to understanding this weight filtration on $\mathbf{N}^\bullet_X$. 
• Let $D_X = \text{supp} \mathfrak{N}_X^\bullet$, and let $i : D_X \hookrightarrow X$ be the closed inclusion. We can then find a smooth, Zariski open dense subset $\mathcal{W} \subseteq D_X$ over which the normalization map restricts to a covering projection [Goresky-MacPherson 1983 [2]]

$$\hat{\pi} : \pi^{-1}(\mathcal{W}) \rightarrow \mathcal{W} \subseteq D_X.$$
The Set-Up

- Let $D_X = \text{supp} \mathcal{N}_X^\bullet$, and let $i : D_X \hookrightarrow X$ be the closed inclusion. We can then find a smooth, Zariski open dense subset $\mathcal{W} \subseteq D_X$ over which the normalization map restricts to a covering projection [Goresky-MacPherson 1983 [2]]

$$\hat{\pi} : \pi^{-1}(\mathcal{W}) \to \mathcal{W} \subseteq D_X.$$

- Let $l : \mathcal{W} \hookrightarrow D_X$ and $m : D_X \setminus \mathcal{W} \hookrightarrow D_X$ denote the respective open and closed inclusion maps. Let $\hat{m} := i \circ m$, $\hat{l} := i \circ l$. 
The Set-Up

• Let $D_X = \text{supp} \mathbf{N}^\bullet_X$, and let $i : D_X \hookrightarrow X$ be the closed inclusion. We can then find a smooth, Zariski open dense subset $\mathcal{W} \subseteq D_X$ over which the normalization map restricts to a covering projection [Goresky-MacPherson 1983 [2]]

$$\hat{\pi} : \pi^{-1}(\mathcal{W}) \to \mathcal{W} \subseteq D_X.$$

• Let $l : \mathcal{W} \hookrightarrow D_X$ and $m : D_X \setminus \mathcal{W} \hookrightarrow D_X$ denote the respective open and closed inclusion maps. Let $\hat{m} := i \circ m$, $\hat{l} := i \circ l$. 

![Diagram](image-url)
Theorem (H., 2018 [5])

Suppose $X$ is a parameterized space. Then, there is an isomorphism

\[
\text{Gr}^W_{n-1} i^* N_X^\bullet \cong I^\bullet_C D_X (\hat{l}^* N_X^\bullet),
\]

so that the short exact sequence of perverse sheaves on $X$

\[
0 \rightarrow m_* p H^0 (m^! i^* N_X^\bullet) \rightarrow i^* N_X^\bullet \rightarrow I^\bullet_C D_X (\hat{l}^* N_X^\bullet) \rightarrow 0
\]

identifies $W_{n-2} i^* N_X^\bullet \cong m_* p H^0 (m^! i^* N_X^\bullet)$. 

Theorem (H.,2018 [5])

Suppose $X$ is a parameterized space. Then, there is an isomorphism
\[
\text{Gr}_{n-1}^W i^* N_X^\bullet \cong IC_{D_X}(\hat{l}^* N_X^\bullet),
\]
so that the short exact sequence of perverse sheaves on $X$
\[
0 \to m_* p H^0(m! i^* N_X^\bullet) \to i^* N_X^\bullet \to IC_{D_X}(\hat{l}^* N_X^\bullet) \to 0
\]
identifies $W_{n-2} i^* N_X^\bullet \cong m_* p H^0(m! i^* N_X^\bullet)$. Here, $IC_{D_X}(\hat{l}^* N_X^\bullet)$ denotes the intermediate extension of the perverse sheaf $\hat{l}^* N_X^\bullet$ to all of $D_X$, and $p H^0(-)$ denotes the 0-th perverse cohomology functor.
Theorem (H., 2018 [5])

Suppose $X$ is a parameterized space. Then, there is an isomorphism
\[ \text{Gr}^W_{n-1} i^* N^\bullet_X \cong \text{IC}^\bullet_{D_X}(\hat{l}^* N^\bullet_X), \]
so that the short exact sequence of perverse sheaves on $X$
\[
0 \to m_* p H^0(m^! i^* N^\bullet_X) \to i^* N^\bullet_X \to \text{IC}^\bullet_{D_X}(\hat{l}^* N^\bullet_X) \to 0
\]
identifies $W_{n-2} i^* N^\bullet_X \cong m_* p H^0(m^! i^* N^\bullet_X)$. Here, $\text{IC}^\bullet_{D_X}(l^* N^\bullet_X)$ denotes the intermediate extension of the perverse sheaf $l^* N^\bullet_X$ to all of $D_X$, and $p H^0(\cdot)$ denotes the 0-th perverse cohomology functor.

Corollary

Suppose $X$ is a parameterized space. Then, there are isomorphisms
\[
\text{Gr}^W_{n-1} \mathbb{Q}^\bullet_X[n] \cong \text{Gr}^W_{n-1} i^* N^\bullet_X \cong i_* \text{IC}^\bullet_{D_X}(\hat{l}^* N^\bullet_X).
\]
Sketch of Theorem

We show that there exists a short exact sequence of perverse sheaves (underlying MHM)

\[ 0 \to \mathfrak{m}^* \mathfrak{p} H_0(m!i^*N^\bullet X) \to i^*N^\bullet X \to IC^\bullet D_X(\hat{l}^*N^\bullet X) \to 0, \]

provided \( \hat{m}^*[-1]N^\bullet X \in \mathfrak{p} D_{\leq 0}(D_X/W) \), called the enhanced support condition along \( D_X/W \).

For surfaces with curve singularities, \( \hat{m}^*[-1]N^\bullet X \in \mathfrak{p} D_{\leq 0}(D_X/W) \) if and only if \( X \) is a parameterized space.

Next, we show \( IC^\bullet D_X(\hat{l}^*N^\bullet X) \) underlies a polarizable Hodge module of weight \( n-1 \). Along \( W \), \( \hat{l}^*N^\bullet X \) underlies a smooth Hodge module with stalks \( H^{−n+1}(N^\bullet X) \approx \tilde{H}_0(K_{\tilde{X}}, \pi^{-1}(p); Q) \), which is a pure Hodge structure of weight 0.

Finally, we prove \( Gr_{W^{n−1}}i^*N^\bullet X \approx IC^\bullet D_X(\hat{l}^*N^\bullet X) \) by using the decomposition by strict support property on \( Gr_{W^{n−1}}i^*N^\bullet X \).
Sketch of Theorem

- We show that there exists a short exact sequence of perverse sheaves (underlying MHM)

\[ 0 \to m_*^p H^0(m^! i^* N^\bullet_X) \to i^* N^\bullet_X \to IC^\bullet_{D_X}(\hat{l}^* N^\bullet_X) \to 0, \]

provided \( \hat{m}^* [-1] N^\bullet_X \in p D^{\leq 0}(D_X \setminus \mathcal{W}) \), called the **enhanced support condition** along \( D_X \setminus \mathcal{W} \).
Sketch of Theorem

- We show that there exists a short exact sequence of perverse sheaves (underlying MHM)

\[ 0 \to m_*^pH^0(m^!i^*N_X^\bullet) \to i^*N_X^\bullet \to \text{IC}^\bullet_{D_X}(\hat{l}^*N_{X}^\bullet) \to 0, \]

provided $\hat{m}^*[-1]N_{X}^\bullet \in pD^{\leq 0}(D_X \setminus \mathcal{W})$, called the enhanced support condition along $D_X \setminus \mathcal{W}$.

- For surfaces with curve singularities, $\hat{m}^*[-1]N_{X}^\bullet \in pD^{\leq 0}(D_X \setminus \mathcal{W})$ if and only if $X$ is a parameterized space.
Sketch of Theorem

• We show that there exists a short exact sequence of perverse sheaves (underlying MHM)

\[ 0 \rightarrow m_* p H^0(m^! i^* N^\bullet_X) \rightarrow i^* N^\bullet_X \rightarrow IC^\bullet_{D_X}(i^* N^\bullet_X) \rightarrow 0, \]

provided \( \hat{m}^*[-1]N^\bullet_X \in p D^{\leq 0}(D_X \setminus W) \), called the enhanced support condition along \( D_X \setminus W \).

• For surfaces with curve singularities, \( \hat{m}^*[-1]N^\bullet_X \in p D^{\leq 0}(D_X \setminus W) \) if and only if \( X \) is a parameterized space.

• Next, we show \( IC^\bullet_{D_X}(i^* N^\bullet_X) \) underlies a polarizable Hodge module of weight \( n - 1 \). Along \( W \), \( i^* N^\bullet_X \) underlies a smooth Hodge module with stalks \( H^{-n+1}(N^\bullet_X)_p \cong \tilde{H}^0(K_{\tilde{X}, \pi^{-1}(p)}; \mathbb{Q}) \), which is a pure Hodge structure of weight 0.
Sketch of Theorem

• We show that there exists a short exact sequence of perverse sheaves (underlying MHM)

\[
0 \rightarrow m_*^p H^0(m_!i^*N_X^\bullet) \rightarrow i^*N_X^\bullet \rightarrow IC_{D_X}(\hat{i}^*N_X^\bullet) \rightarrow 0,
\]

provided \( \hat{m}^*[-1]N_X^\bullet \in pD^{\leq 0}(D_X \setminus \mathcal{W}) \), called the **enhanced support condition** along \( D_X \setminus \mathcal{W} \).

• For surfaces with curve singularities, \( \hat{m}^*[-1]N_X^\bullet \in pD^{\leq 0}(D_X \setminus \mathcal{W}) \) if and only if \( X \) is a parameterized space.

• Next, we show \( IC_{D_X}(\hat{i}^*N_X^\bullet) \) underlies a **polarizable Hodge module** of weight \( n - 1 \). Along \( \mathcal{W} \), \( \hat{i}^*N_X^\bullet \) underlies a smooth Hodge module with stalks \( H^{-n+1}(N_X^\bullet)_p \cong \tilde{H}^0(K_{\tilde{X},\pi^{-1}(p)}; \mathbb{Q}) \), which is a pure Hodge structure of weight 0.

• Finally, we prove \( Gr_{n-1}^W i^*N_X^\bullet \cong IC_{D_X}(\hat{i}^*N_X^\bullet) \) by using the decomposition by strict support property on \( Gr_{n-1}^W i^*N_X^\bullet \).
When $X$ is a purely 2-dimensional parameterized space, we can determine the rest of the weight filtration.
When $X$ is a purely 2-dimensional parameterized space, we can determine the rest of the weight filtration.

* The previous theorem tells us

$$W_0 \mathbb{Q}_X^\bullet [2] = i_*^p H^0 (m^! i^* N_X^\bullet).$$

* When $X$ is a surface, $D_X \setminus \mathcal{W}$ is purely 0-dimensional; hence $p H^0 (m^!) = H^0 (m^!)$ is just ordinary cohomology:
When $X$ is a purely 2-dimensional parameterized space, we can determine the rest of the weight filtration.

- The previous theorem tells us
  \[ W_0 \mathbb{Q}^\bullet_X[2] = i_* p H^0(m^! i^* N^\bullet_X). \]

- When $X$ is a surface, $D_X \setminus \mathcal{W}$ is purely 0-dimensional; hence $p H^0(m^!) = H^0(m^!)$ is just ordinary cohomology:
  \[ p H^0(m^! i^* N^\bullet_X) = H^0(m^! i^* N^\bullet_X) = H^0(D_X, D_X \setminus \{0\}; i^* N^\bullet_X). \]
When $X$ is a purely 2-dimensional parameterized space, we can determine the rest of the weight filtration.

- The previous theorem tells us
  \[ W_0 \mathbb{Q}_X^\bullet[2] = i_* \mathcal{P}^0(\mathfrak{m}^! i^* \mathcal{N}_X^\bullet). \]

- When $X$ is a surface, $D_X \setminus \mathcal{W}$ is purely 0-dimensional; hence $\mathcal{P}^0(\mathfrak{m}^!) = H^0(\mathfrak{m}^!)$ is just ordinary cohomology:
  \[ \mathcal{P}^0(\mathfrak{m}^! i^* \mathcal{N}_X^\bullet) = H^0(\mathfrak{m}^! i^* \mathcal{N}_X^\bullet) = \mathbb{H}^0(D_X, D_X \setminus \{0\}; i^* \mathcal{N}_X^\bullet). \]

- Taking the long exact sequence in relative hypercohomology with coefficients in $i^* \mathcal{N}_X^\bullet$ then gives the following result:
Theorem (H., 2018 [5])

Suppose $X$ is a parameterized surface. Then, $\mathbb{Q}_X^\bullet[2]$ has weights $\geq 0$, and

$$\text{Gr}_0^W \mathbb{Q}_X^\bullet[2] \cong V_{\{0\}}$$

is a perverse sheaf concentrated on a single point, i.e., a finite-dimensional $\mathbb{Q}$-vector space,
Parameterized Surfaces, II

**Theorem (H., 2018 [5])**

Suppose $X$ is a parameterized surface. Then, $\mathbb{Q}_X^\bullet[2]$ has weights $\geq 0$, and

$$\text{Gr}^W_0 \mathbb{Q}_X^\bullet[2] \cong V^\bullet_{\{0\}}$$

is a perverse sheaf concentrated on a single point, i.e., a finite-dimensional $\mathbb{Q}$-vector space, of dimension

$$\dim V = 1 - |\pi^{-1}(0)| + \sum_C \dim \ker \{\text{id} - h_C\},$$
Theorem (H.,2018 [5])

Suppose $X$ is a parameterized surface. Then, $\mathbb{Q}_X^\bullet[2]$ has weights $\geq 0$, and

$$\text{Gr}_0^W \mathbb{Q}_X^\bullet[2] \cong V_{\{0\}}$$

is a perverse sheaf concentrated on a single point, i.e., a finite-dimensional $\mathbb{Q}$-vector space, of dimension

$$\dim V = 1 - |\pi^{-1}(0)| + \sum_C \dim \ker\{\text{id} - h_C\},$$

where $\{C\}$ is the collection of irreducible components of $D_X$ at $0$, and for each component $C$, $h_C$ is the (internal) monodromy operator on the local system $H^{-1}(N_X^\bullet)_{|C\setminus\{0\}}$. Note that $|\pi^{-1}(0)|$ is, of course, equal to the number of irreducible components of $X$ at $0$. 
Examples: Maps $(\mathbb{C}^2, S) \rightarrow (\mathbb{C}^3, 0)$

- Our main (and easiest) source of examples for parameterized surfaces come from finite, generically one-to-one maps $\pi : (\mathbb{C}^2, S) \rightarrow (\mathbb{C}^3, 0)$, where $S = \pi^{-1}(0)$. Such examples trivially yield parameterized surfaces $\text{im} \pi = V(f)$ for some function $f \in \mathcal{O}_{\mathbb{C}^3, 0}$. In particular, we give special attention to finitely-determined maps.
Examples: Maps $(\mathbb{C}^2, S) \rightarrow (\mathbb{C}^3, 0)$

- Our main (and easiest) source of examples for parameterized surfaces come from finite, generically one-to-one maps $\pi : (\mathbb{C}^2, S) \rightarrow (\mathbb{C}^3, 0)$, where $S = \pi^{-1}(0)$. Such examples trivially yield parameterized surfaces $\text{im} \pi = V(f)$ for some function $f \in \mathcal{O}_{\mathbb{C}^3,0}$. In particular, we give special attention to finitely-determined maps.

- For these cases, the stalk of the local system $\hat{\pi}^* \mathcal{N}_{V(f)}^\bullet$ at any point $p \in \Sigma f$ is isomorphic to $\mathbb{Q}$, since the transverse type of $f$ along $\Sigma f$ is that of a Morse function.
Examples: Maps $(\mathbb{C}^2, S) \rightarrow (\mathbb{C}^3, 0)$

- Our main (and easiest) source of examples for parameterized surfaces come from finite, generically one-to-one maps $\pi : (\mathbb{C}^2, S) \rightarrow (\mathbb{C}^3, 0)$, where $S = \pi^{-1}(0)$. Such examples trivially yield parameterized surfaces $\text{im} \pi = V(f)$ for some function $f \in \mathcal{O}_{\mathbb{C}^3, 0}$. In particular, we give special attention to \textbf{finitely-determined maps}.

- For these cases, the stalk of the local system $\hat{\mathcal{N}}^\bullet_{V(f)}$ at any point $p \in \Sigma f$ is isomorphic to $\mathbb{Q}$, since the transverse type of $f$ along $\Sigma f$ is that of a Morse function.

- Hence, the invariant sections $\ker\{\text{id} - h_C\}$ are all either 0 or $\mathbb{Q}$ as well, with monodromy operator $h_C = \pm 1$. 
Examples

The Whitney Umbrella $V(f) = V(y^2 - x^3 - zx^2)$ has $\Sigma f = V(x, y)$, and smooth normalization given by $\pi(u, t) = (u^2 - t, u(u^2 - t), t)$. Since the internal monodromy operator around the origin is multiplication by $-1$, we have

$$\text{Gr}^W_0 \mathbb{Q}^\bullet_{V(f)}[2] = 0.$$
Examples

\[ V(g) = V(y^2 - x^3 - z^2x^2) \] has \[ \Sigma g = V(x, y) \], and smooth normalization given by \[ \pi(u, t) = (u^2 - t^2, u(u^2 - t^2), t) \]. Since the internal monodromy operator is the identity, we have

\[ \text{Gr}^{W}_0 \, \mathbb{Q}^*_{V(g)}[2] \cong \mathbb{Q}^*_\{0\} \].

\[ \tilde{X} \]

\[ D_x \]
Examples

The triple point singularity $V(h) = V(xyz)$ has
$\Sigma h = V(x, y) \cup V(y, z) \cup V(x, z)$, and smooth normalization given by
pulling apart the three coordinate planes. The internal monodromy
operators around the origin are all the identity operator, so we find

$$\text{Gr}^W_0 Q^*_{V(h)}[2] \cong Q^*_{\{0\}}.$$
Topological Invariance

**Theorem (H., 2019 [5])**

Suppose \((X, 0_X), (Y, 0_Y)\) are two surface germs, \(Y\) parameterized, and let \(\alpha : (X, 0_X) \rightarrow (Y, 0_Y)\) be a homeomorphism. Then, there are isomorphisms

\[
W_i \mathbb{Q}_X[2] \cong \alpha^* W_i \mathbb{Q}_Y[2]
\]

for \(i = 0, 1, 2\).
Theorem (H., 2019 [5])

Suppose \((X, 0_X), (Y, 0_Y)\) are two surface germs, \(Y\) parameterized, and let \(\alpha : (X, 0_X) \rightarrow (Y, 0_Y)\) be a homeomorphism. Then, there are isomorphisms

\[
W_i \mathbb{Q}_X^\bullet [2] \cong \alpha^* W_i \mathbb{Q}_Y^\bullet [2]
\]

for \(i = 0, 1, 2\).

We are motivated by the invariance of \(\text{IC}^\bullet_X\) \((i = 2\) above\); this is a well-known topological invariant of any complex analytic space.
Sketch of Proof, 1

- When $X$ is purely 2-dimensional, perversity of $\mathbb{Q}^\bullet_X[2]$ is topological: $\mathbb{Q}^\bullet_X[2]$ is perverse if and only if the real link $K_{X,p}$ is connected for all $p \in X$. This follows easily from the cosupport condition on $\mathbb{Q}^\bullet_X[2]$. 
• When $X$ is purely 2-dimensional, perversity of $Q^\bullet_X[2]$ is topological: $Q^\bullet_X[2]$ is perverse if and only if the real link $K_{X,p}$ is connected for all $p \in X$. This follows easily from the cosupport condition on $Q^\bullet_X[2]$.

• Hence, for any homeomorphism $\alpha : X \to Y$, the surjection $Q^\bullet_Y[2] \to IC_Y^\bullet$ is sent by $\alpha^*$ to the surjection $Q^\bullet_X[2] \to IC_X^\bullet$. Thus, $\alpha^*N_Y^\bullet \cong N_X^\bullet$ by definition.
Sketch of Proof, I

• When $X$ is purely 2-dimensional, perversity of $\mathbb{Q}^\bullet_X[2]$ is topological: $\mathbb{Q}^\bullet_X[2]$ is perverse if and only if the real link $K_{X,p}$ is connected for all $p \in X$. This follows easily from the cosupport condition on $\mathbb{Q}^\bullet_X[2]$.

• Hence, for any homeomorphism $\alpha : X \to Y$, the surjection $\mathbb{Q}^\bullet_Y[2] \to \mathbf{IC}^\bullet_Y$ is sent by $\alpha^*$ to the surjection $\mathbb{Q}^\bullet_X[2] \to \mathbf{IC}^\bullet_X$. Thus, $\alpha^* N^\bullet_Y \cong N^\bullet_X$ by definition.

• The support of $N^\bullet_X$ is now topological as well, by the previous isomorphism; that is, $D_X$ is mapped homeomorphically onto $D_Y$ by $\alpha$. 

Hence, we have isomorphisms of perverse sheaves (underlying polarizable variations of HS of weight zero):

\[
N^\bullet_X|_\mathcal{W} \cong (\alpha^* N^\bullet_Y)|_\mathcal{W} \cong \alpha^* \left( N^\bullet_Y|_{\alpha(\mathcal{W})} \right).
\]
Hence, we have isomorphisms of perverse sheaves (underlying polarizable variations of HS of weight zero):

\[ N_X^\bullet |_{\mathcal{W}} \cong (\alpha^* N_Y^\bullet) |_{\mathcal{W}} \cong \alpha^* \left( N_Y^\bullet |_{\alpha(\mathcal{W})} \right). \]

Finally, the claim follows from the isomorphisms

\[ \text{Gr}_{1}^{W} \mathbb{Q}_X^\bullet [2] \cong \text{IC}_{D_X}^\bullet (N_X^\bullet |_{\mathcal{W}}) \cong \alpha^* \text{IC}_{D_Y}^\bullet (N_Y^\bullet |_{\alpha(\mathcal{W})}), \]
Hence, we have isomorphisms of perverse sheaves (underlying polarizable variations of HS of weight zero):

\[ N_X^\bullet |_W \cong (\alpha^* N_Y^\bullet) |_W \cong \alpha^* N_Y^\bullet |_{\alpha(W)}. \]

Finally, the claim follows from the isomorphisms

\[ \text{Gr}^W_1 \mathbb{Q}_X[2] \cong \text{IC}_{D_X}^\bullet (N_X^\bullet |_W) \cong \alpha^* \text{IC}_{D_Y}^\bullet (N_Y^\bullet |_{\alpha(W)}), \]

using the characterization of the intermediate extension as the unique (up to iso) perverse sheaf extending \( N_X^\bullet |_W \) to all of \( D_X \) with no perverse sub or quotient objects contained in \( \{0\} \).
Consequences: Vanishing Cycles

When $X = V(f)$ is a hypersurface in $\mathbb{C}^{n+1}$, Massey showed in [7] that there is an isomorphism

$$N^\bullet_{V(f)} \cong \ker \{ \text{id} - \tilde{T}_f \},$$

where $\tilde{T}_f$ is the Milnor monodromy automorphism on the vanishing cycles $\phi_f[-1] \mathbb{Z}^\bullet_{\mathbb{C}^{n+1}}[n + 1]$. 
Consequences: Vanishing Cycles

When $X = V(f)$ is a hypersurface in $\mathbb{C}^{n+1}$, Massey showed in [7] that there is an isomorphism

$$N^\bullet_{V(f)} \cong \ker\{\text{id} - \tilde{T}_f\},$$

where $\tilde{T}_f$ is the Milnor monodromy automorphism on the vanishing cycles $\phi_f[-1]Z^\bullet_{\mathbb{C}^{n+1}}[n + 1]$.

As a mixed Hodge module, if we let $N = \log T_u$ denote the logarithm of the unipotent part of the monodromy operator on $\phi_{f,1}[-1]Q^\bullet_{\mathbb{C}^{n+1}}[n + 1]$,
When $X = V(f)$ is a hypersurface in $\mathbb{C}^{n+1}$, Massey showed in [7] that there is an isomorphism

$$N_{V(f)}^\bullet \cong \ker \{ \text{id} - \tilde{T}_f \},$$

where $\tilde{T}_f$ is the Milnor monodromy automorphism on the vanishing cycles $\phi_f[-1]\mathbb{Z}_{\mathbb{C}_{n+1}}[n+1]$.

As a mixed Hodge module, if we let $N = \log T_u$ denote the logarithm of the unipotent part of the monodromy operator on $\phi_f,1[-1]\mathbb{Q}_{\mathbb{C}_{n+1}}[n+1]$, one can further show that

$$N_{V(f)}^\bullet \cong \ker N(1),$$

where (1) denotes the Tate twist operator.
Consequences: Vanishing Cycles

Corollary

For parameterized surfaces $V(f)$ in $\mathbb{C}^3$, the graded pieces of the weight filtration on $\phi_{f,1}[-1]Q_{\mathbb{C}^3}[3]$ are as follows:
Consequences: Vanishing Cycles

Corollary

For parameterized surfaces $V(f)$ in $\mathbb{C}^3$, the graded pieces of the weight filtration on $\phi_{f,1}[−1]Q^\bullet\mathbb{C}^3[3]$ are as follows:

$$\text{Gr}^W_{3+k} \phi_{f,1}[−1]Q^\bullet\mathbb{C}^3[3] \cong \begin{cases} \text{IC}^\bullet_{\Sigma f}(\hat{l}^*\mathcal{N}_{V(f)}(-1)), & \text{if } k = 0, \\ V^\bullet_{\{0\}}(-1), & \text{if } k = -1, 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $V$ is a $\mathbb{Q}$-vector space of dimension

$$\dim_{\mathbb{Q}} V = 1 - |\pi^{-1}(0)| + \sum_C \dim \ker\{\text{id} - h_C\}.$$
Consequences: Hodge Filtration

- Since we know $N_X^\bullet$ underlies a mixed Hodge module, there exists a filtered $\mathcal{D}$-module $(N_X^H, F^\bullet N_X^H)$ on $X$ with $\text{DR}(N_X^H) \cong N_X^\bullet \otimes \mathbb{C}$.
Consequences: Hodge Filtration

- Since we know $N_X^\bullet$ underlies a mixed Hodge module, there exists a filtered $\mathcal{D}$-module $(N_X^H, F_\bullet N_X^H)$ on $X$ with $\text{DR}(N_X^H) \cong N_X^\bullet \otimes \mathbb{C}$.

- Moreover, when $X$ is a reduced hypersurface in a complex manifold $U$, by the fundamental SES defining $N_X^\bullet$ as a perverse sheaf, there exists a corresponding SES of filtered $\mathcal{D}$-modules defining $N_X^H$:

$$0 \to \mathcal{L}(X, U) \to H^1_{[X]}(\mathcal{O}_U) \to N_X^H \to 0$$

where $H^1_{[X]}(\mathcal{O}_U)$ is the algebraic local cohomology module, and $\mathcal{L}(X, U)$ is the intersection cohomology $\mathcal{D}$-module.
Consequences: Hodge Filtration

- Since we know $N_X^\bullet$ underlies a mixed Hodge module, there exists a filtered $\mathcal{D}$-module $(N^H_X, F \cdot N^H_X)$ on $X$ with $\text{DR}(N^H_X) \cong N_X^\bullet \otimes \mathbb{C}$.

- Moreover, when $X$ is a reduced hypersurface in a complex manifold $U$, by the fundamental SES defining $N_X^\bullet$ as a perverse sheaf, there exists a corresponding SES of filtered $\mathcal{D}$-modules defining $N^H_X$:

$$0 \rightarrow \mathcal{L}(X, U) \rightarrow H^1_{[X]}(\mathcal{O}_U) \rightarrow N^H_X \rightarrow 0$$

where $H^1_{[X]}(\mathcal{O}_U)$ is the algebraic local cohomology module, and $\mathcal{L}(X, U)$ is the intersection cohomology $\mathcal{D}$-module.

- The Hodge filtration on $H^1_{[X]}(\mathcal{O}_U)$ then induces a good filtration on $N^H_X$. 
Consequences: Hodge Filtration

- It is well-known that $\mathcal{H}^1_{[X]}(\mathcal{O}_U)$ is described by the short exact sequence

$$0 \to \mathcal{O}_U \to \mathcal{O}_U(*X) \to H^1_{[X]}(\mathcal{O}_U) \to 0,$$

where the Hodge filtration on $\mathcal{O}_U^*(X)$ (since $U$ is smooth) is

$\text{Gr}_p\mathcal{O}_U = 0$ for $p \neq 0$, and $F_p\mathcal{O}_U((p+1)X) \otimes \mathcal{I}_p(X)$,

where the $\mathcal{I}_p(X)$ are the Hodge ideals of $X$ of M. Mustată and M. Popa.

- Thus, $F_pH^1_{[X]}(\mathcal{O}_U) = \mathcal{O}_U((p+1)X) \otimes \mathcal{I}_p(X)$, and $F_p\mathcal{N}_{H^1X} = \text{im}\{F_pH^1_{[X]}(\mathcal{O}_U) \to \mathcal{N}_{H^1X}\}$. 
Consequences: Hodge Filtration

- It is well-known that $H^1_{[X]}(\mathcal{O}_U)$ is described by the short exact sequence
  
  $$0 \to \mathcal{O}_U \to \mathcal{O}_U(\ast X) \to H^1_{[X]}(\mathcal{O}_U) \to 0,$$

- where the Hodge filtration on $\mathcal{O}_U$ (since $U$ is smooth) is $\text{Gr}^F_p \mathcal{O}_U = 0$ for $p \neq 0$, and
  
  $$F_p \mathcal{O}_U(\ast X) = \mathcal{O}_U((p + 1)X) \otimes I_p(X),$$

  where the $I_p(X)$ are the Hodge ideals of $X$ of M. Mustață and M. Popa.
Consequences: Hodge Filtration

- It is well-known that $H^1_{[X]}(\mathcal{O}_U)$ is described by the short exact sequence

$$0 \rightarrow \mathcal{O}_U \rightarrow \mathcal{O}_U(*X) \rightarrow H^1_{[X]}(\mathcal{O}_U) \rightarrow 0,$$

- where the Hodge filtration on $\mathcal{O}_U$ (since $\mathcal{U}$ is smooth) is $\text{Gr}^F_p \mathcal{O}_U = 0$ for $p \neq 0$, and

$$F_p \mathcal{O}_U(*X) = \mathcal{O}_U((p+1)X) \otimes l_p(X),$$

where the $l_p(X)$ are the Hodge ideals of $X$ of M. Mustață and M. Popa.

- Thus, $F_p H^1_{[X]}(\mathcal{O}_U) = \frac{\mathcal{O}_U((p+1)X) \otimes l_p(X)}{F_p \mathcal{O}_U}$, and

$$F_p N^H_X = \text{im}\{F_p H^1_{[X]}(\mathcal{O}_U) \rightarrow N^H_X\}.$$
Consequences: Sing. of Maps

Given a finite, generically one-to-one map \( \pi : (\mathbb{C}^2, S) \rightarrow (\mathbb{C}^3, 0) \), topological invariance of the weight filtration on \( \mathbb{Q}_{\text{im } \pi}^\bullet [2] \) implies that we can also interpret the weight filtration as an \( \mathcal{A} \)-invariant of the map \( \pi \):
Consequences: Sing. of Maps

Given a finite, generically one-to-one map \( \pi : (\mathbb{C}^2, S) \rightarrow (\mathbb{C}^3, 0) \), topological invariance of the weight filtration on \( \mathbb{Q}^\bullet \text{im } \pi [2] \) implies that we can also interpret the weight filtration as an \( \mathcal{A} \)-invariant of the map \( \pi \):

- For any pair of local biholomorphisms of the domain and codomain \( (L, R) \in \mathcal{A}, \pi \circ R^{-1} \) is still a normalization of the image of \( \pi \), and
Consequences: Sing. of Maps

Given a finite, generically one-to-one map $\pi : (\mathbb{C}^2, S) \to (\mathbb{C}^3, 0)$, topological invariance of the weight filtration on $\mathbb{Q}_{\text{im } \pi}^\bullet [2]$ implies that we can also interpret the weight filtration as an $\mathcal{A}$-invariant of the map $\pi$:

- For any pair of local biholomorphisms of the domain and codomain $(L, R) \in \mathcal{A}$, $\pi \circ R^{-1}$ is still a normalization of the image of $\pi$, and

- $(\text{im } \pi, 0)$ is homeomorphic to $(\text{im}(L \circ \pi \circ R^{-1}), 0)$. 
Consequences: Sing. of Maps

- Since the levels $W_i \mathcal{Q}^\bullet_{V(f)}[2]$ give a perverse composition series of $\mathcal{Q}^\bullet_{V(f)}[2]$, we find that

$$\mu_I(\pi) = (\Gamma_1^{f,L} \cdot V(L))_0 = w_0(V(f)) + \lambda^0_{\mathcal{I}_D^\bullet_{V(f)}}(\hat{\ast} \mathcal{N}^\bullet_{V(f)}, L(0)) + \lambda^0_{\mathcal{I}_C^\bullet_{V(f)}, L(0)},$$

where $\Gamma_1^{f,L}$ is the relative polar curve of $f$ with respect to a (generic) linear form $L$, $w_0(V(f))$ is the dimension of $Gr_{W_0} \mathcal{Q}^\bullet_{V(f)}[2]$, and $\lambda^0_{\mathcal{I}_C^\bullet_{V(f)}, L(0)}$ denotes the 0-dimensional characteristic polar multiplicities of $\mathcal{I}_C^\bullet_{V(f)}$ with respect to $L$ at 0.

Since all of these numbers on the RHS are non-negative, this means that the complex link of $V(f)$ at 0 is contractible if and only if these three numbers are zero.
Consequences: Sing. of Maps

• Since the levels $W_i \mathbb{Q}^\bullet_{V(f)}[2]$ give a perverse composition series of $\mathbb{Q}^\bullet_{V(f)}[2]$, we find that

$$\mu_l(\pi) = (\Gamma_{f,L}^1 \cdot \mathcal{V}(L))_0 = w_0(V(f)) + \lambda^0_{\IC_{D_{V(f)}}(\hat{*}N_{V(f)}),L}(0) + \lambda^0_{\IC_{V(f)},L}(0),$$

where $\Gamma_{f,L}^1$ is the relative polar curve of $f$ with respect to a (generic) linear form $L$, $w_0(V(f))$ is the dimension of $\Gr^W_0 \mathbb{Q}^\bullet_{V(f)}[2]$, and $\lambda^0_{\bullet,L}(0)$ denotes the 0-dimensional characteristic polar multiplicities of $\IC_{D_{V(f)}}(\hat{*}N_{V(f)})$ and $\IC_{V(f)}$ with respect to $L$ at $0$. 

Since all of these numbers on the RHS are non-negative, this means that the complex link of $\mathcal{V}(f)$ at $0$ is contractible if and only if these three numbers are zero.
Consequences: Sing. of Maps

• Since the levels $W_i Q^\bullet_{V(f)}[2]$ give a perverse composition series of $Q^\bullet_{V(f)}[2]$, we find that

$$\mu_L(\pi) = (\Gamma^1_{f,L} \cdot V(L))_0 = w_0(V(f)) + \lambda^0_{IC_{D_V(f)}(\hat{\pi}^* N^\bullet_{V(f)}), L}(0) + \lambda^0_{IC_{V(f)}, L}(0),$$

where $\Gamma^1_{f,L}$ is the relative polar curve of $f$ with respect to a (generic) linear form $L$, $w_0(V(f))$ is the dimension of $Gr^W_0 Q^\bullet_{V(f)}[2]$, and $\lambda^0_{*, L}(0)$ denotes the 0-dimensional characteristic polar multiplicities of $IC_{D_V(f)}(\hat{\pi}^* N^\bullet_{V(f)})$ and $IC_{V(f)}$ with respect to $L$ at $0$.

• Since all of these numbers on the RHS are non-negative, this means that the complex link of $V(f)$ at $0$ is contractible if and only if these three numbers are zero.


