Deformation Formulas for Parametrizable Hypersurfaces

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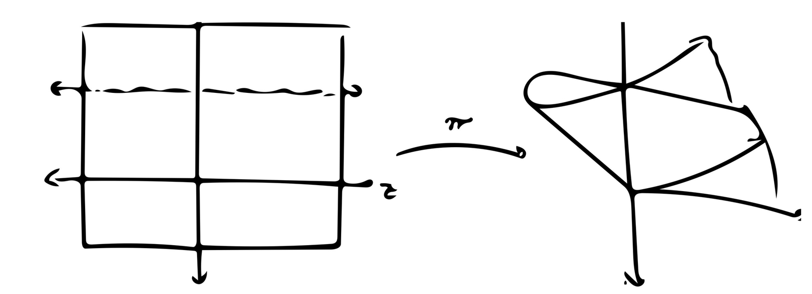

Abstract

We investigate one-parameter deformations of functions on affine space which define parameterizable hypersurfaces. With the assumption of isolated polar activity at the origin, we are able to completely express the Lê numbers of the special fiber in terms of the Lê numbers of the generic fiber and the characteristic polar multiplicities of the comparison, a perverse sheaf naturally associated to any reduced complex analytic space on which the constant sheaf $\mathbb{Q}^\bullet_X[\dim X]$ is perverse.

This generalizes the classical formula for the Milnor number of a plane curve in terms of double points as well as Mond's image Milnor number. We also recover results of Gaffney and Bobadilla using this framework. We obtain similar deformation formulas for maps from $\mathbb{C}^2$ to $\mathbb{C}^3$, and provide an ansatz for obtaining deformation formulas for all dimensions within Mather's nice dimensions.

Milnor's Double Point Formula [7],[4]

Suppose $(V(f_0), \mathbf{0}) \subseteq (\mathbb{C}^2, \mathbf{0})$ is plane curve singularity in $\mathbb{C}^2$, with r irreducible components at the origin. Then, by a well-known result of Milnor (1968), the Milnor number $\mu_{\mathbf{0}}(g_0)$ is related to the number of double points δ which occur in a generic (stable) deformation of f_0 by

$$\mu_{\mathbf{0}}(f_0) = 2\delta - r + 1.$$

We achieve a quick proof of this result in [4] using the techniques below. We now wish to generalize this formula to deformations of hypersurfaces with codimension-one singularities.

In particular, we consider the case where $V(f)$ is **parameterized**. This is to say that there is a finite, generically one-to-one morphism $\pi : (\mathcal{W}, S) \to (\mathcal{U}, \mathbf{0})$ with $\operatorname{im}\pi = V(f)$ (this is equivalent to $V(f)$ having a smooth normalization).

Fundamental Short Exact Sequence [2]

When $X = V(f)$ is a hypersurface in $\mathbb{C}^{n+1}$ (or, more generally any reduced analytic space on which $\mathbb{Q}^\bullet_X[n]$ is perverse, like an LCI), there is a canonical short exact sequence of perverse sheaves relating the constant sheaf and intersection cohomology:

$$0 \to \mathbf{N}^\bullet_{V(f)} \to \mathbb{Q}^\bullet_{V(f)}[n] \to \mathbf{IC}^\bullet_{V(f)} \to 0 \tag{1}$$

on X. As $\mathbb{Q}^\bullet_{V(f)}[n]$ and $\mathbf{IC}^\bullet_{Vf()}$ are, essentially, the two fundamental perverse sheaves on the space, we refer to 1 as the **fundamental short exact sequence**. This short exact sequence, and the perverse sheaf $\mathbf{N}^\bullet_{V(f)}$ in particular, have examined recently in several papers by the author and D. Massey in the case where the normalization of X is smooth [4] and [1], and where the normalization is a rational homology manifold [2]. In these papers, we refer to $\mathbf{N}^\bullet_{V(f)}$ as the **multiple-point complex** of the normalization, as it naturally encodes the data about the image multiple-points of the normalization. The multiple-point complex governs when both $V(f)$ and its normalization are **rational homology manifolds** [2].

Conservation of Number [1]

Suppose that $\pi : (\mathbb{C} \times \widetilde{V(f_0)}, \{0\} \times S) \to (V(f), \mathbf{0})$ is a one-parameter unfolding of a parameterized hypersurface $\operatorname{im}\pi_0 = V(f_0)$. Suppose further that $\mathbf{z} = (z_1, \cdots, z_n)$ is chosen such that $\mathbf{z}$ is an IPA-tuple for $f_0 = f_{|_{V(t)}}$ at $\mathbf{0}$. Then, the following formulas hold for the Lê numbers of f_0 with respect to $\mathbf{z}$ at $\mathbf{0}$: for $0 < |t_0| \ll \epsilon \ll 1$, $0 \le i \le n-2$:

$$\lambda^i_{f_0,\mathbf{z}}(\mathbf{0}) + \lambda^i_{\mathbf{N}^\bullet_{V(f_0)},\mathbf{z}}(\mathbf{0}) = \sum_{p \in B_\epsilon \cap V(t-t_0, z_1, z_2, \cdots, z_i)} \left(\lambda^i_{f_{t_0},\mathbf{z}}(p) + \lambda^i_{\mathbf{N}^\bullet_{V(f_{t_0})},\mathbf{z}}(p) \right).$$

Since the Lê numbers of f are the same as the characteristic polar multiplicities of the vanishing cycles $\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]$, **this result suggests a conservation of number property with the characteristic polar multiplicities of the comparison complex and the vanishing cycles.**

For a precise definition of characteristic polar multiplicities, see [6]; for deformations with isolated polar activity (IPA-deformations and IPA-tuples), see [5].

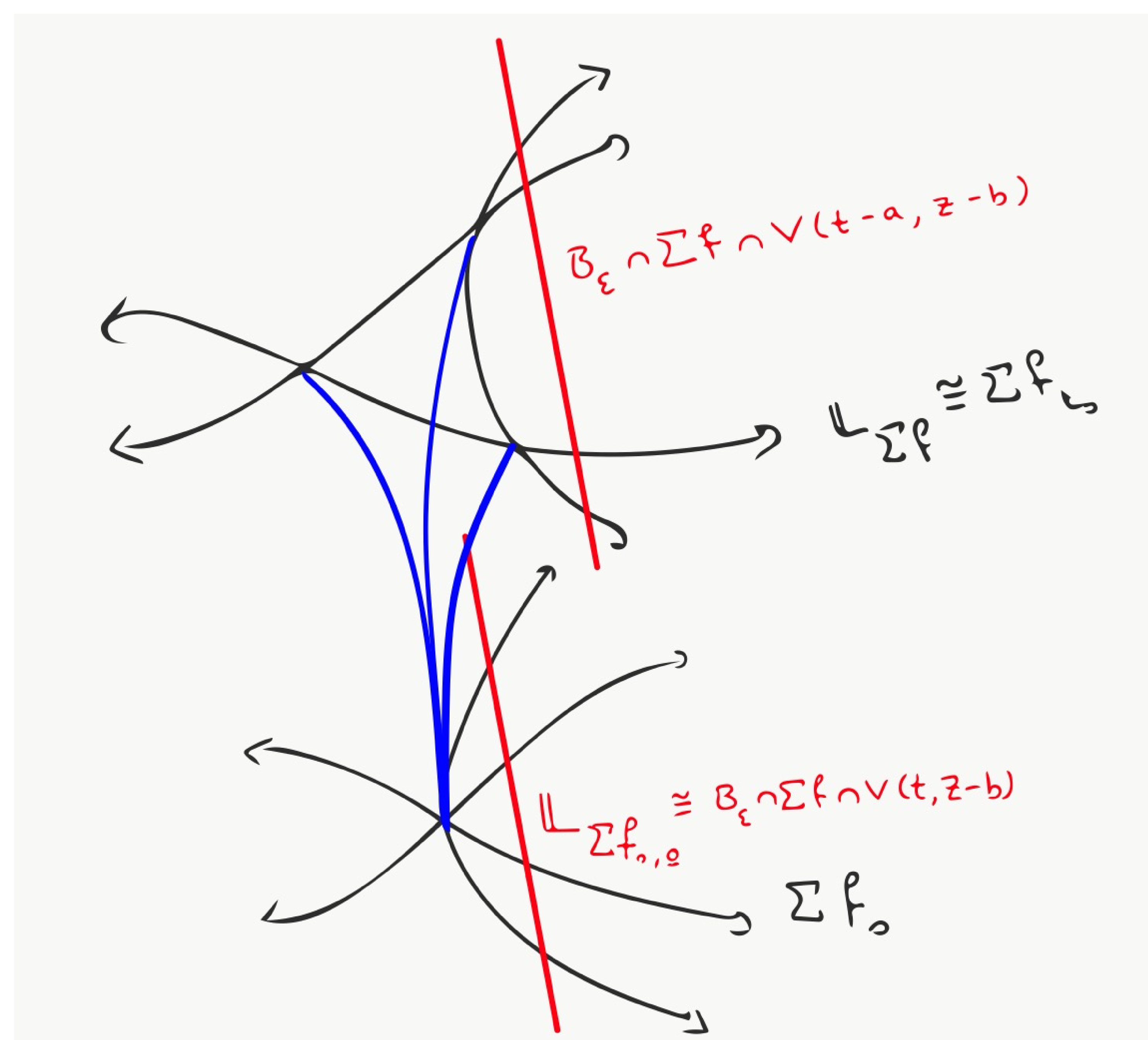

Fig. 1: Deforming a Parameterized Surface.

Finitely-Determined Maps $\mathbb{C}^2 \to \mathbb{C}^3$ [1]

Let $\pi_0 : (\mathbb{C}^2, S) \to (\mathbb{C}^3, \mathbf{0})$ be a finitely-determined map germ parameterizing a surface $V(f_0) \subseteq \mathbb{C}^3$, and let T, C, and δ, denote the number of triple points, cross caps, and A_1-singularities, respectively, appearing in a stabilization of π_0. Then, the following equality holds:

$$|\pi_0^{-1}(\mathbf{0})| - 1 = -C + T + \delta + \chi(\mathbb{L}_{\Sigma f,\mathbf{0}}),$$

where $\mathbb{L}_{\Sigma f,\mathbf{0}}$ denotes the Milnor fiber of the unfolding parameter of such a stabilization $\operatorname{im}\pi = V(f)$, restricted to the singular locus of f (that is, the **complex link** of Σf at $\mathbf{0}$).

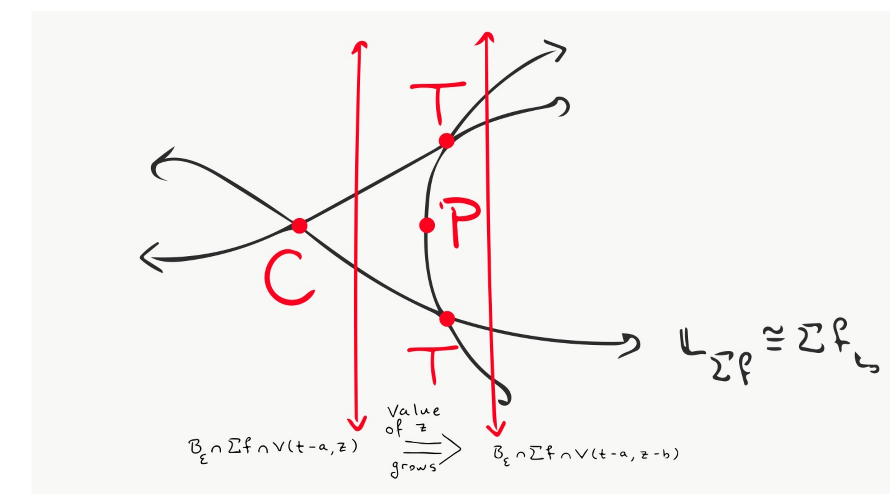

Fig. 2: Using IPA-deformations to calculate $\lambda^1_{\mathbb{Q}_{V(f)},\mathbf{z}}(\mathbf{0})$

The Weight Filtration [3]

Moreover, we determine that, for parameterized surfaces $V(f) \subseteq \mathbb{C}^3$, the weight filtration on $\mathbb{Q}^\bullet_{V(f)}[2]$ is concentrated in degrees $[0, 2]$, and that the weight levels $W_i\mathbb{Q}^\bullet_{V(f)}[2]$ are local **topological invariants** of $V(f)$ at $\mathbf{0}$.

Then,

$$\operatorname{Gr}^W_0 \mathbb{Q}^\bullet_{V(f)}[2] \cong W_0\mathbb{Q}^\bullet_{V(f)}[2] \cong V^\bullet_{\{\mathbf{0}\}}$$

is a perverse sheaf concentrated on a single point, i.e., a finite-dimensional $\mathbb{Q}$-vector space, of dimension

$$\dim V = 1 - |\pi^{-1}(\mathbf{0})| + \sum_C \dim\ker\{\operatorname{Id} - h_C\},$$

where $\{C\}$ is the collection of irreducible components of Σf at $\mathbf{0}$, and for each component C, h_C is the (internal) monodromy operator on the local system $H^{-1}(\mathbf{N}^\bullet_{V(f)})_{|_{C \setminus \{\mathbf{0}\}}}$. Note that $|\pi^{-1}(\mathbf{0})|$ is, of course, equal to the number of irreducible components of $V(f)$ at $\mathbf{0}$.

References

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