Hypersurface Normalizations and Numerical Invariants

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Abstract of Dissertation

We define a new perverse sheaf, the comparison complex, naturally associated to any locally reduced complex analytic space X on which the (shifted) constant sheaf $\mathbb{Q}^\bullet_X[\dim X]$ is perverse. In the hypersurface case, this complex is isomorphic to the perverse eigenspace of the eigenvalue one for the Milnor monodromy action on the vanishing cycles; we also examine how the characteristic polar multiplicities of this complex behave in certain one-parameter families of deformations of hypersurfaces with codimension-one singularities, and generalize a classical formula for the Milnor number of a plane curve singularities in terms of double-points. In general, the vanishing of the cohomology sheaves of the comparison complex provide a criterion for determining if the normalization of the space X is a rational homology manifold. When the normalization is a rational homology manifold, we can also compute several terms in the weight filtration of the constant sheaf $\mathbb{Q}^\bullet_X[\dim X]$ in those cases for which this perverse sheaf underlies a mixed Hodge module. In the surface case $V(f) \subseteq \mathbb{C}^3$, this produces a new numerical invariant, the weight zero part of the constant sheaf, which is a perverse sheaf concentrated on a single point.

We then prove two special cases of a conjecture of Javier Fernández de Bobadilla for hypersurfaces with 1-dimensional critical loci (Corollary 4.2.0.2 and Theorem 4.3.0.2). We do this via a new numerical invariant for such hypersurfaces, called the beta invariant, first defined and explored by the Massey in 2014. The beta invariant is an algebraically calculable invariant of the local ambient topological-type of the hypersurface, and the vanishing of the beta invariant is equivalent to the hypotheses of Bobadilla's conjecture. Bobadilla's Conjecture is related to a more well-known conjecture by Lê Dũng Tráng (Conjecture 4.0.0.1) regarding the equsingularity of parameterized surfaces in $\mathbb{C}^3$.

Table of Contents

Disclaimer

I hereby declare that the work in this thesis is that of the candidate alone, except where indicated in the text, and as described below.

Introduction

The main focus of the author's research is the local topology of complex analytic spaces with arbitrary singularities. Largely, this research is concerned with the use of perverse sheaves and derived category techniques in extending classical results for isolated singularities in affine or projective space to arbitrarily singular complex analytic spaces.

The main object of study that threads together the work in this dissertation is a perverse sheaf called the **comparison complex**, first defined by the author and David Massey in [23] (where we originally referred to it as the **multiple-point complex**, for reasons that will become clear in Formula 1.4), and subsequently studied in several papers by the author in [20], [21], [22], and Massey in [39]. This perverse sheaf, denoted $\mathbf{N}^\bullet_X$, is defined on any pure-dimensional (locally reduced) complex analytic space X for which the constant sheaf $\mathbb{Q}^\bullet_X[\dim X]$ is perverse.

On such a space, there are two "fundamental" perverse sheaves: the constant sheaf $\mathbb{Q}^\bullet_X[\dim X]$, and the intersection cohomology complex $\mathbf{I}^\bullet_X$ with constant coefficients, with a natural surjective map $\mathbb{Q}^\bullet_X[\dim X] \to \mathbf{I}^\bullet_X \to 0$, where, in a certain sense, $\mathbf{I}^\bullet_X$ detects the singularities of X. Then, the comparison complex $\mathbf{N}^\bullet_X$ "compares" these two complexes in a manner analogous to the way the vanishing cycles compares the constant sheaf on a (possibly singular) hypersurface $V(f)$ with a nearby smooth fiber of f. The success of the vanishing cycles in understanding the the topology of complex analytic spaces cannot be overstated, and it is therefore natural to hope that $\mathbf{N}^\bullet_X$ will have similarly wide application. The simplest class of singular spaces we can analyze with $\mathbf{N}^\bullet_X$ are those we call parameterized spaces.

For such spaces, we obtain a generalization of MIlnor's classical formula for plane curve singularities $V(f_0) \subseteq \mathbb{C}^2$ in terms of double points appearing in a stable deformation $\mu_{\mathbf{0}}(f_0) = 2\delta - r + 1$ (where δ is the number of double points, and r is the number of irreducible components of $V(f_0)$ at $\mathbf{0}$). For higher-dimensional parameterized hypersurfaces $V(f_0)$, this generalization is expressed in terms of the Lê numbers of f_0 and f_{t_0}, and the characteristic polar multiplicities of $\mathbf{N}^\bullet_{V(f_0)}$ and $\mathbf{N}^\bullet_{V(f_{t_0})}$:

$$\lambda^0_{f_0,\mathbf{z}}(\mathbf{0}) = -\lambda^0_{\mathbf{N}^\bullet_{V(f_0)},\mathbf{z}}(\mathbf{0}) + \sum_{p \in B_\epsilon \cap V(t-t_0)} \left(\lambda^0_{f_{t_0},\mathbf{z}}(p) + \lambda^0_{\mathbf{N}^\bullet_{V(f_{t_0})},\mathbf{z}}(p) \right).$$

While these numerical invariants allow us to compute information about these higher-dimensional singluarities, they do also include contributions from points on the absolute polar curve, which arise from the choice of coordinates (see Theorem 2.4.0.2 and Theorem 2.4.0.7)

Throughout Chapter 1,2, and 3, we will be primarily concerned **parameterized spaces** (Definition 1.1.1.7). These spaces are locally reduced, and have codimension-one singularities. While this might normally make the topology of these spaces hard to analyze, our approach using perverse sheaves allows us to make some headway. Particularly, these spaces are interesting because (using $\mathbb{Z}$ coefficients) the constant sheaf $\mathbb{Z}^\bullet_X[\dim X]$ is perverse, and intersection cohomology with constant coefficients is of the form $\mathbf{I}^\bullet_X \cong \pi_* \mathbb{Z}^\bullet_{\widetilde{X}}[\dim X]$, where $\widetilde{X} \xrightarrow{\pi} X$ is the normalization of X. With just these properties, we can conclude many things.

In Chapter 1, we introduce parameterized spaces, the fundamental short exact sequence of perverse sheaves on X, and Theorem 1.1.1.4, which is our main criterion for determining if a space is parameterized. The main tool throughout this chapter is the comparison complex $\mathbf{N}^\bullet_X$, a perverse sheaf naturally defined on any pure-dimensional, locally reduced space, which has a particularly nice form for parameterized spaces (which we have also called the multiple-point complex on such spaces). We conclude Chapter 1 by investigating Milnor fibers of functions defined on parameterized spaces, and give formulas for computing the cohomology of these Milnor fibers in terms of the (hyper)cohomology of the comparison complex (Theorem 1.3.1.4). Using the techniques developed in Section 1.3, we first encounter and re-prove Milnor's formula for the Milnor number of a plane curve singularity in terms of the number of double-points appearing in a stable deformation of the curve (Theorem 1.3.2.4). The results from this chapter (unless otherwise specified) are from [21] by the author, and [23] by the author and David Massey.

In Chapter 2, our main goal is to generalize Milnor's formula (Theorem 1.3.2.4) to higher dimensions using deformations of parameterized hypersurfaces $V(f)$. To do this, we apply the theory of **deformations with isolated polar activity** (IPA-deformations, Section 2.1, originally developed by David Massey in [41]) as the "correct" way to deform a parameterized space so that we can apply the same methods used in our proof of Milnor's formula from the curve case. The numerical invariants involved in this theory are the Lê numbers of the function defining the hypersurface $V(f)$, and the characteristic polar multiplicities of the comparison complex $\mathbf{N}^\bullet_{V(f)}$, are the "correct" numbers to keep track of when deforming parameterized hypersurfaces. Finally, in Section 2.4, we prove Theorem 2.4.0.2, which generalizes Milnor's theorem to deformations of parameterized hypersurfaces of arbitrary dimension. We give particular examples of the formulas we obtain for deformations of parameterized surfaces in $\mathbb{C}^3$ in Theorem 2.4.0.7 and Corollary 2.4.0.8 . Such formulas exist for deformations of parameterized hypersurfaces $V(f)$ in $\mathbb{C}^{n+1}$, provided we stay in Mather's "nice dimensions" $n < 15$. The results from this chapter (unless otherwise specified) are from the preprint [20] by the author.

In Chapter 3, we investigate the question: "when is $\mathbf{N}^\bullet_X$ a semi-simple perverse sheaf, so that $\mathbb{Q}^\bullet_X[n]$ is an extension of semi-simple perverse sheaves?" This question was posed to us by a referee of [21], and led us to understand the structure of $\mathbf{N}^\bullet_X$ as a mixed Hodge module on X. It is well-known that, locally, $\mathbb{Q}^\bullet_X[n]$ underlies a mixed Hodge module of weight $\leq n$ on X, with weight n graded piece isomorphic to the intersection cohomology complex $\mathbf{I}^\bullet_X$ with constant $\mathbb{Q}$ coefficients. In this chapter we identify the weight $n-1$ graded piece $\mathrm{Gr}^W_{n-1}\,\mathbb{Q}^\bullet_X[n]$ in the case where X is a parameterized space (Theorem 3.2.0.6). We then give concrete computations of the perverse sheaf $W_0\mathbb{Q}^\bullet_{V(f)}[2]$ in the case where $X = V(f)$ is a parameterized surface in $\mathbb{C}^3$ (Theorem 3.3.0.1). We conclude the chapter with some connections to the work of David Massey regarding the eigenspaces of the Milnor monodromy (Section 3.4), and the work of Morihiko Saito in the complex algebraic setting (Section 3.5). The results from this chapter (unless otherwise specified) are from the preprint [22] by the author.

In Chapter 4, we prove two special cases of a conjecture of Javier Fernández de Bobadilla for hypersurfaces with 1-dimensional critical loci (Corollary 4.2.0.2 and Theorem 4.3.0.2). We do this via a new numerical invariant for such hypersurfaces, called the beta invariant, first defined and explored by the Massey in 2014. The beta invariant is an algebraically calculable invariant of the local, ambient topological-type of the hypersurface, and the vanishing of the beta invariant is equivalent to the hypotheses of Bobadilla's conjecture. Bobadilla's Conjecture is related to a more well-known conjecture by Lê Dũng Tráng (Conjecture 4.0.0.1) regarding the equsingularity of parameterized surfaces in $\mathbb{C}^3$, which was our original motivation for approaching Bobadilla's Conjecture. The results from this chapter (unless otherwise specified) are from the paper [24] by the author and David Massey.

Chapter 1

Parameterized Spaces

1.1 The Fundamental Short Exact Sequence

Let $\mathcal{U}$ be a connected open neighborhood of the origin in $\mathbb{C}^N$, and let $X \subseteq \mathcal{U}$ be a reduced complex analytic space containing $\mathbf{0}$ of pure dimension n, on which the (shifted) constant sheaf $\mathbb{Z}^\bullet_X[n]$ is perverse (e.g., if X is a local complete intersection).

There is then a surjection of perverse sheaves $\mathbb{Z}^\bullet_X[n] \to \mathbf{I}^\bullet_X \to 0$, where $\mathbf{I}^\bullet_X$ is the intersection cohomology complex on X with constant $\mathbb{Z}$ coefficients. This follows from the fact that $\mathbf{I}^\bullet_X$ is also the intermediate extension of the local system $\mathbb{Z}^\bullet_{X\backslash\Sigma X}[n]$ to all of X (where ΣX denotes the singular locus of X), and therefore has no perverse quotient objects with support contained in ΣX. Since $\mathbb{Z}^\bullet_X[n] \to \mathbf{I}^\bullet_X$ is an isomorphism when restricted to $X\backslash\Sigma X$, the cokernel of this morphism is zero.

Remark 1.1.0.1. If one works with $\mathbb{Q}$ coefficients, $\mathbb{Q}^\bullet_X[n]$ is still perverse, and $\mathbf{I}^\bullet_X$ (with $\mathbb{Q}$ coefficients) is a semi-simple object in the category of perverse sheaves on X with no perverse sub or quotient objects with support contained in ΣX. Since the cokernel of the natural morphism $\mathbb{Q}^\bullet_X[n] \to \mathbf{I}^\bullet_X$ has support contained in ΣX, the natural morphism must be surjective.

Since the category of perverse sheaves on X is Abelian, there is a perverse sheaf $\mathbf{N}_X^\bullet$ on X such that

$$0 \to \mathbf{N}_X^\bullet \to \mathbb{Z}_X^\bullet[n] \to \mathbf{I}_X^\bullet \to 0 \tag{1.1}$$

is a short exact sequence of perverse sheaves. When $\mathbb{Z}_X^\bullet[n]$ is perverse, this sheaf and $\mathbf{I}_X^\bullet$ are, essentially, the two fundamental perverse sheaves on X. For this reason, we make the following definition.

Definition 1.1.0.2. *We refer to (1.1) as the* ***fundamental short exact sequence*** *of X.*

Definition 1.1.0.3. *We refer to the perverse sheaf $\mathbf{N}_X^\bullet$ as the* ***comparison complex*** *on X.*

The comparison complex was first defined and explored by the author and David Massey in [23], and subsequently studied in several papers by the author [20],[21],[22] and Massey [39]. The comparison complex $\mathbf{N}_X^\bullet$ and the fundamental short exact sequence are the main objects of study in Chapter 1, Chapter 2, and Chapter 3.

1.1.1 The Fundamental Short Exact Sequence and The Normalization

Let $\pi : \widetilde{X} \to X$ be the normalization of X. The map π is finite and generically one-to-one; in particular, π is a small map, in the sense of Goresky and MacPherson:

Definition 1.1.1.1 (Goresky-MacPherson,[16]). *A proper, surjective morphism of varieties $f : Y \to Z$ is* ***small*** *if, for all $k > 0$,*

$$\operatorname{codim}_{\mathbb{C}}\{z \in Z \mid \dim_{\mathbb{C}} f^{-1}(z) = k\} > 2k.$$

For the purposes of this paper, the most important property of small maps is the following.

Theorem 1.1.1.2 (Goresky-MacPherson,[16]). *Suppose $f : Y \to Z$ is a small map. Then, $f_*\mathbf{I}_Y^\bullet \cong \mathbf{I}_Z^\bullet$.*

Consequently, the normalization map allows one to reinterpret the fundamental short exact sequence (1.1) as

$$0 \to \mathbf{N}^\bullet_X \to \mathbb{Z}^\bullet_X[n] \to \pi_*\mathbf{I}^\bullet_{\widetilde{X}} \to 0. \tag{1.2}$$

Note that taking stalk cohomology at $p \in X$ of the fundamental short exact sequence yields the short exact sequence

$$0 \to \mathbb{Z} \to H^{-n}(\pi_*\mathbf{I}^\bullet_{\widetilde{X}})_p \to H^{-n+1}(\mathbf{N}^\bullet_X)_p \to 0, \tag{1.3}$$

and isomorphisms $H^k(\pi_*\mathbf{I}^\bullet_{\widetilde{X}})_p \cong H^{k+1}(\mathbf{N}^\bullet_X)_p$ for $-n+1 \leq k \leq -1$.

Remark 1.1.1.3. The reader may be wondering why the morphism $\mathbb{Z}^\bullet_X[n] \to \pi_*\mathbf{I}^\bullet_{\widetilde{X}}$ in (1.2) (which we will sometimes call Δ) has a non-zero kernel in the category of perverse sheaves; it may seem as though the short exact sequence (1.3) is the correct interpretation. After all, on the level of stalks, this morphism is the diagonal inclusion map

$$\mathbb{Z} \xrightarrow{H^{-n}(\Delta)_p} \bigoplus_{q\in\pi^{-1}(p)} \mathbb{H}^{-n}(K_{\widetilde{X},q}; \mathbf{I}^\bullet_{\widetilde{X}}) \cong \bigoplus_{q\in\pi^{-1}(p)} \mathbb{Z};$$

it may seem as though Δ should have a non-trivial cokernel, not kernel.

It is true that there is a complex of sheaves $\mathbf{C}^\bullet$ and a distinguished triangle in the derived category

$$\mathbb{Z}^\bullet_X[n] \xrightarrow{\Delta} \pi_*\mathbf{I}^\bullet_{\widetilde{X}} \to \mathbf{C}^\bullet \xrightarrow{[1]} \mathbb{Z}^\bullet_X[n]$$

in which the stalk cohomology of $\mathbf{C}^\bullet$ is non-zero only in degrees greater than or equal to $-n$ and, in degree $-n$, is isomorphic to the cokernel of map induced on the stalks by Δ. However, the complex $\mathbf{C}^\bullet$ is **not** perverse; it is supported on a set of dimension less than or equal to $n-1$ and has non-zero cohomology in degree $-n$.

However, we can "turn" the triangle to obtain a distinguished triangle

$$\mathbf{C}^\bullet[-1] \to \mathbb{Z}^\bullet_X[n] \to \pi_*\mathbf{I}^\bullet_{\widetilde{X}}[n] \xrightarrow{[1]} \mathbf{C}^\bullet,$$

where $\mathbf{C}^\bullet[-1]$ is, in fact, perverse. Thus, in the Abelian category of perverse sheaves $\mathbf{N}^\bullet_X := \mathbf{C}^\bullet[-1]$ is the kernel of the morphism Δ.

Over $\mathbb{Q}$, one notices immediately that $\mathbb{Q}^\bullet_X[n] \cong \mathbf{I}^\bullet_X$ if and only if $\mathbf{N}^\bullet_X = 0$; that is, the space X is an **rational homology manifold** (or, a **$\mathbb{Q}$-homology manifold**) precisely when the complex $\mathbf{N}^\bullet_X$ vanishes (for this criterion, see for example [4], [40]). It is then natural to ask that, given the normalization $\widetilde{X}$ of X and the resulting fundamental short exact sequence, is there a similar result relating $\mathbf{N}^\bullet_X$ to whether or not $\widetilde{X}$ is a $\mathbb{Q}$-homology manifold? This turns out to be true, which we prove as the main result of [21].

Theorem 1.1.1.4. *The normalization $\widetilde{X}$ of X is a $\mathbb{Q}$-homology manifold if and only if $\mathbf{N}^\bullet_X$ has stalk cohomology concentrated in degree $-n+1$; i.e., for all $p \in X$, $H^k(\mathbf{N}^\bullet_X)_p$ is non-zero only possibly when $k = -n+1$.*

The proof of Theorem 1.1.1.4 relies on the following well-known lemma.

Lemma 1.1.1.5. *Let X be a complex analytic space of pure dimension n. Then, for $p \in X$, the rank of $H^{-n}(\mathbf{I}^\bullet_X)_p$ is equal to the number of irreducible components of X at p.*

Proof. This result is well-known to experts, see e.g. Theorem 1G (pg. 74) of [71], or Theorem 4 (pg. 217) [32] □

With this in mind, we prove Theorem 1.1.1.4

Proof. ($\Longrightarrow$) Suppose that $\widetilde{X}$ is a $\mathbb{Q}$-homology manifold, and let $p \in X$ be arbitrary. Since $\widetilde{X}$ is a $\mathbb{Q}$-homology manifold, $\mathbb{Q}_{\widetilde{X}}[n] \cong \mathbf{I}^\bullet_{\widetilde{X}}$ in $D^b_c(\widetilde{X})$, from which it follows $H^k(\mathbf{N}^\bullet_X)_p = 0$ for $k \neq -n+1$ by the above isomorphisms.

($\Longleftarrow$) Suppose that, for all $p \in X$, $H^k(\mathbf{N}^\bullet_X)_p \neq 0$ only possibly when $k = -n+1$. We wish to show that the natural morphism $\mathbb{Q}_{\widetilde{X}}[n] \to \mathbf{I}^\bullet_{\widetilde{X}}$ is an isomorphism in $D^b_c(\widetilde{X})$.

There is still the short exact sequence

$$0 \to \mathbb{Q} \to H^{-n}(\pi_* \mathbf{I}^\bullet_{\widetilde{X}})_p \to H^{-n+1}(\mathbf{N}^\bullet_X)_p \to 0$$

and $H^k(\pi_*\mathbf{I}^\bullet_{\widetilde{X}})_p = 0$ for $k \neq -n$, since $H^k(\pi_*\mathbf{I}^\bullet_{\widetilde{X}})_p \cong H^{k+1}(\mathbf{N}^\bullet_X)_p$ for all $p \in X$ and $-n+1 \leq k \leq -1$. In degree $-n$, we have

$$H^{-n}(\pi_*\mathbf{I}^\bullet_{\widetilde{X}})_p \cong \bigoplus_{q \in \pi^{-1}(p)} H^{-n}(\mathbf{I}^\bullet_{\widetilde{X}})_q.$$

This then implies that, for all $q \in \widetilde{X}$, $H^k(\mathbf{I}^\bullet_{\widetilde{X}})_q = 0$ for $k \neq -n$. Our goal is to calculate this stalk cohomology in degree $-n$. Since $\widetilde{X}$ is normal, and thus locally irreducible, it follows by Lemma 1.1.1.5 that $H^{-n}(\mathbf{I}^\bullet_{\widetilde{X}})_q \cong \mathbb{Q}$ for all $q \in \widetilde{X}$.

Finally, we claim that the natural morphism $\mathbb{Q}^\bullet_{\widetilde{X}}[n] \to \mathbf{I}^\bullet_{\widetilde{X}}$ is an isomorphism in $D^b_c(\widetilde{X})$. In stalk cohomology at any point $q \in \widetilde{X}$, both $H^k(\mathbb{Q}^\bullet_{\widetilde{X}}[n])_q$ and $H^k(\mathbf{I}^\bullet_{\widetilde{X}})_q$ are non-zero only in degree $k = -n$, with stalks isomorphic to $\mathbb{Q}$. Consequently, the natural morphism is an isomorphism in $D^b_c(\widetilde{X})$ provided that the morphism

$$\mathbb{Q} \cong H^{-n}(\mathbb{Q}^\bullet_{\widetilde{X}}[n])_q \to H^{-n}(\mathbf{I}^\bullet_{\widetilde{X}})_q \cong \mathbb{Q}$$

is not the zero morphism. But this is just the "diagonal" morphism from a single copy of $\mathbb{Q}$ to the number of connected components of $\widetilde{X}\backslash\{p\}$, which is clearly non-zero. Thus, $\widetilde{X}$ is a $\mathbb{Q}$-homology manifold. $\square$

Remark 1.1.1.6. If the normalization $\widetilde{X}$ is smooth, then of course one has that $\mathbb{Z}^\bullet_{\widetilde{X}}[n] \cong \mathbf{I}^\bullet_{\widetilde{X}}$, in which case $\mathbf{N}^\bullet_X$ is concentrated in degree $-n+1$ by Theorem 1.1.1.4, even over $\mathbb{Z}$. This was the original context in which the author and Massey investigated $\mathbf{N}^\bullet_X$ in [23], where we referred to it as the **multiple-point complex**, for reasons that will become clear in Formula 1.4.

One really does need to use $\mathbb{Q}$ coefficients in general for Theorem 1.1.1.4 to hold; the failure of this statement for $\mathbb{Z}$ coefficients is demonstrated in Subsection 1.1.2.

We now arrive at the central definition of the chapter (and, in fact, of this entire thesis): the notion of a parameterized space.

Definition 1.1.1.7. *A reduced, purely n-dimensional complex analytic space X on which the complex $\mathbb{Z}^\bullet_X[n]$ is perverse is called a* ***parameterized space*** *if the normalization of X is either a $\mathbb{Q}$-homology manifold (when using $\mathbb{Q}$ coefficients) or smooth (when using $\mathbb{Z}$ coefficients). Clearly, results that hold with $\mathbb{Z}$ coefficients hold also when using $\mathbb{Q}$ coefficients, but not vice-versa.*

Corollary 1.1.1.8. *When X is a parameterized space, the costalk cohomology of $\mathbf{N}^\bullet_X$ is given by, for all $x \in X$,*

$$H^k(j_x^! \mathbf{N}^\bullet_X) \cong \begin{cases} \widetilde{H}^{n+k-1}(K_{X,x}; \mathbb{Z}), & \text{if } 0 \leq k \leq n-1; \\ 0, & \text{otherwise.} \end{cases}$$

where $j_x : \{x\} \hookrightarrow X$ is the inclusion of a point, and $K_{X,x}$ is the ***real link*** *of X at x, i.e., the intersection of X with a sufficiently small sphere centered at x.*

When X is a parameterized space, the short exact sequence

$$0 \to \mathbb{Z} \to H^{-n}(\pi_* \mathbf{I}^\bullet_{\widetilde{X}})_p \to H^{-n+1}(\mathbf{N}^\bullet_X)_p \to 0$$

allows us to identify, given Lemma 1.1.1.5, that

$$m(p) := \operatorname{rank}_{\mathbb{Z}} H^{-n+1}(\mathbf{N}^\bullet_X)_p = |\pi^{-1}(p)| - 1. \tag{1.4}$$

Consequently, we conclude that the support of $\mathbf{N}^\bullet_X$ is none other than the **image multiple-point set** of the morphism π, which we denote by D; precisely, we have

$$D := \operatorname{supp} \mathbf{N}^\bullet_X = \overline{\{p \in X \mid |\pi^{-1}(p)| > 1\}}. \tag{1.5}$$

For this reason, we have referred to the perverse sheaf $\mathbf{N}^\bullet_X$ as the **multiple-point complex** of X (or, of the morphism π, as we do in [20] and [23]). It is immediate from the fundamental short exact sequence that one always has the inclusion $D \subseteq \Sigma X$.

In such cases (see, e.g., Subsection 1.1.2), it is useful to partition X into subsets $X_k = m^{-1}(k)$ for $k \geq 1$; clearly, one has

$$D = \overline{\bigcup_{k>1} X_k}. \tag{1.6}$$

Finally, since D is the support of a perverse sheaf which, on an open dense subset of D, has non-zero stalk cohomology only in degree $-n+1$, it follows that D is purely $(n-1)$-dimensional.

1.1.2 A Trivial, Non-Trivial Example

We consider the following example of the normalization of a surface X with one-dimensional singularity in $\mathbb{C}^3$, which nicely illustrates the content of Theorem 1.1.1.4, and why one wants to use $\mathbb{Q}$ coefficients.

Let $f(x,y,z) = xz^2 - y^2(y+x^3)$, so that $X = V(f) \subseteq \mathbb{C}^3$ has critical locus $\Sigma f = V(y,z)$. Then, if we let $\widetilde{X} = V(u^2 - x(y+x^3), uy - xz, uz - y(y+x^3)) \subseteq \mathbb{C}^4$, the projection map $\pi : \widetilde{X} \to X$ forgetting the variable u is the normalization of X.

It is easy to check that $\Sigma\widetilde{X} = V(x,y,z,u)$, and

$$\pi^{-1}(\Sigma f) = V(u^2 - x^4, y, z).$$

It then follows that $X_k = \emptyset$ if $k > 2$, and $X_2 = V(y,z)\backslash\{\mathbf{0}\}$, so that

$$\operatorname{supp} \mathbf{N}^\bullet_X = V(y,z) = \Sigma f.$$

For $p \in X$,

$$H^{-2}(\pi_*\mathbf{I}^\bullet_{\widetilde{X}})_p \cong \bigoplus_{q \in \pi^{-1}(p)} H^{-2}(\mathbf{I}^\bullet_{\widetilde{X}})_q \quad (\dagger 4.1) \tag{1.7}$$

$$\tag{1.8}$$

But $\pi^{-1}(p) \subseteq \widetilde{X}\backslash\Sigma\widetilde{X}$, and $\left(\mathbf{I}^\bullet_{\widetilde{X}}\right)_{|_{\pi^{-1}(p)}} \cong \left(\mathbb{Q}^\bullet_{\widetilde{X}}[2]\right)_{|_{\pi^{-1}(p)}}$, so from (1.7), it follows that

$$H^{-2}(\pi_*\mathbf{I}^\bullet_{\widetilde{X}})_p \cong \mathbb{Q}^2.$$

Similarly, since $\left(\mathbf{I}^\bullet_{\widetilde{X}}\right)_{\widetilde{X}\backslash\Sigma\widetilde{X}} \cong \mathbb{Q}^\bullet_{\widetilde{X}\backslash\Sigma\widetilde{X}}[2]$, it follows that

$$H^0(\mathbf{N}^\bullet_X)_p \cong H^{-1}(\pi_*\mathbf{I}^\bullet_{\widetilde{X}})_p = 0.$$

When $p = \mathbf{0}$, we find

$$H^k(\mathbf{I}^\bullet_{\widetilde{X}})_{\mathbf{0}} \cong \begin{cases} \mathbb{H}^k(K_{\widetilde{X},\mathbf{0}}; \mathbf{I}^\bullet_{\widetilde{X}}), & \text{if } k \leq -1 \\ 0, & \text{if } k > -1 \end{cases}$$

Since $\widetilde{X}$ has an isolated singularity at the origin in $\mathbb{C}^4$, we further have

$$\mathbb{H}^k(K_{\widetilde{X},\mathbf{0}}; \mathbf{I}^\bullet_{\widetilde{X}}) \cong H^{k+2}(K_{\widetilde{X},\mathbf{0}}, \mathbb{Q}).$$

For $0 < \epsilon \ll 1$, the sphere S_ϵ transversely intersects $\widetilde{X}$ near $\mathbf{0}$, so the real link $K_{\widetilde{X},\mathbf{0}} = \widetilde{X} \cap S_\epsilon$ is a compact, orientable, smooth manifold of (real) dimension 3. We are interested in computing the two integral cohomology groups $H^0(K_{\widetilde{X},\mathbf{0}}; \mathbb{Q})$ and $H^1(K_{\widetilde{X},\mathbf{0}}; \mathbb{Q})$.

Because $K_{\widetilde{X},\mathbf{0}}$ is also connected, we can apply Poincaré duality to find $H^0(K_{\widetilde{X},\mathbf{0}}; \mathbb{Q}) \cong \mathbb{Q}$.

Consider the standard parameterization of the twisted cubic $\nu : \mathbb{P}^1 \to \mathbb{P}^3$ via

$$\nu([s:t]) = [s^3 : st^2 : t^3 : s^2t] = [x : y : z : u]$$

which lifts to a map $\nu : \mathbb{C}^2 \to \mathbb{C}^4$, parameterizing the affine cone over the twisted cubic, i.e., the normalization $\widetilde{X} = V(u^2 - xy, uy - xz, uz - y^2)$. Then, we claim that ν is a 3-to-1 covering map away from the origin. Clearly, since ν parameterizes $\widetilde{X}$, we see that ν is a surjective local diffeomorphism onto $\nu(\mathbb{C}^2) = \widetilde{X}$.

Suppose that $\nu(s,t) = \nu(s',t')$. Then, we must have $s^3 = (s')^3$ and $t^3 = (t')^3$, so that there are cube roots of unity η and ω for which $s = \eta s'$ and $t = \omega t'$. But then,

$$s^2t = (s')^2(t') = \eta^2\omega s^2 t,$$

so either $\eta^2\omega = 1$, or $st = 0$. Since η and ω are both cube roots of unity, if $\eta^2\omega = 1$, then $\eta = \omega$. Additionally, note that $st = 0$ implies $(s,t) = \mathbf{0}$. It then follows that ν is 3-to-1 away from the origin.

Consider then the (real analytic) function

$$r(x,y,z,u) = |x|^2 + 3|y|^2 + |z|^2 + 3|u|^2$$

on $\mathbb{C}^4$; r is proper, transversally intersects $\widetilde{X}$ away from $\mathbf{0}$, and $\widetilde{X} \cap r^{-1}[0,\epsilon)$ gives a fundamental system of neighborhoods of the origin in $\widetilde{X}$. Consequently, $\widetilde{X} \cap r^{-1}(\epsilon)$ gives, up to homotopy, the real link $K_{\widetilde{X},\mathbf{0}}$ (see Section 2.A [33]). The composition $r(\nu(s,t))$ then gives:

$$\begin{aligned} r(\nu(s,t)) &= |s^3|^2 + 3|st^2| + |t^3|^2 + 3|s^2t|^2 \\ &= |s|^6 + 3|s|^4|t|^2 + 3|s|^2|t|^2 + |t|^6 \\ &= \left(|s|^2 + |t|^2\right)^3 = \epsilon, \end{aligned}$$

provided that $|s|^2 + |t|^2 = \sqrt[3]{\epsilon}$; that is, ν maps the 3-sphere in $\mathbb{C}^2$ 3-to-1 onto the real link $K_{\widetilde{X},\mathbf{0}}$. Since the 3-sphere is simply-connected, it is the universal cover of $K_{\widetilde{X},\mathbf{0}}$. The group of deck transformations given by multiplying (s,t) by a cube root of unity then yields the isomorphism $\pi_1(K_{\widetilde{X},\mathbf{0}}) \cong \mathbb{Z}/3\mathbb{Z}$. Thus, $H_1(K_{\widetilde{X},\mathbf{0}};\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$.

By again applying Poincaré duality, we find $H^2(K_{\widetilde{X},\mathbf{0}};\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$ as well. By the Universal Coefficient theorem for cohomology, we then have $H_2(K_{\widetilde{X},\mathbf{0}};\mathbb{Z}) = 0$ so that $H^1(K_{\widetilde{X},\mathbf{0}};\mathbb{Z}) = 0$ by Poincaré duality. Using $\mathbb{Q}$ coefficients, this implies:

$$H^k(K_{\widetilde{X},p};\mathbb{Q}) \cong \begin{cases} \mathbb{Q}, & \text{if } k = 0,3 \\ 0, & \text{else} \end{cases}$$

for all $p \in \widetilde{X}$, so that Y is a $\mathbb{Q}$-homology manifold.

Equivalently, we find:

$$H^k(\mathbf{N}^\bullet_X)_p \cong \begin{cases} \mathbb{Q}, & \text{if } k = -1 \text{ and } p \in \Sigma f \backslash \{\mathbf{0}\} \\ 0, & \text{if } k \neq -1,\ p \in \Sigma f \end{cases}$$

i.e., $\mathbf{N}^\bullet_X$ has stalk cohomology concentrated in degree -1.

It is not hard to show that the monodromy of the local system $H^{-1}(\mathbf{N}^\bullet_X)_{|\Sigma f \backslash \{\mathbf{0}\}}$ is trivial: for $p = (e^{it},0,0) \in V(y,z) = \Sigma f$, the preimage consists of the two points $\pi^{-1}(p) = \{\pm(e^{2it}, e^{it}, 0, 0)\}$, and varying t from 0 to 2π, we see that the internal monodromy of the fiber $\pi^{-1}(p)$ exchanges the two points after a half rotation around the origin, and is the identity morphism after a full rotation around the origin.

Consequently, $\mathbf{N}^\bullet_{X|_{\Sigma f}}$ is isomorphic to the extension by zero of the constant sheaf on $\Sigma f \backslash \{\mathbf{0}\}$. That is, if $j : \Sigma f \backslash \{\mathbf{0}\} \hookrightarrow \Sigma f$ is the open inclusion, then $\mathbf{N}^\bullet_{X|_{\Sigma f}} \cong j_! \mathbb{Q}^\bullet_{\Sigma f \backslash \{\mathbf{0}\}}[1]$. In particular, we see that $\mathbf{N}^\bullet_X$ is not semi-simple as a perverse sheaf on X (see Chapter 3).

To compare with Corollary 3.2.0.7, this failure to be a semi-simple perverse sheaf can be detected by the presence of the weight zero part $W_0\mathbf{N}^\bullet_X \cong \mathbb{Q}^\bullet_{\{\mathbf{0}\}} \neq \mathbf{0}$.

1.2 Parameterized Hypersurfaces

Recently, David Massey has shown that, if $X = V(f)$ is a hypersurface, $\mathbf{N}^\bullet_{V(f)} = \ker\{\mathrm{id} - \widetilde{T}_f\}$ is the perverse eigenspace of the eigenvalue 1 of the monodromy action on $\phi_f[-1]\mathbb{Q}^\bullet_{\mathcal{U}}[n+1]$, where $\mathcal{U}$ is an open neighborhood of the origin in $\mathbb{C}^{n+1}$[39]. Here, ϕ_f denotes the functor of vanishing cycles.

In general, it is **not the case** that, given a morphism of perverse sheaves, the cohomology of the stalk of the (perverse) kernel is isomorphic to the kernel of the cohomology on the stalks; that is, there may exist points $p \in \Sigma f$ such that

$$H^k(\ker\{\mathrm{id} - \widetilde{T}_f\})_p \ncong \ker\{\mathrm{id} - \widetilde{T}^k_{f,p}\}.$$

However, this isomorphism **does hold** in degree $-\dim_{\mathbf{0}} \Sigma f = -n+1$ for all $p \in \Sigma f$ (See Lemma 5.1 of [39]):

Proposition 1.2.0.1. *Suppose $V(f)$ is a parameterized space. Then, the following isomorphisms hold for all $p \in \Sigma f$:*

$$H^k(\ker\{\mathrm{id} - \widetilde{T}_f\})_p \cong \begin{cases} \ker\ \{\mathrm{id} - \widetilde{T}^{-n+1}_{f,p}\}, & \text{if } k = -n+1; \\ 0, & \text{if } k \neq -n+1. \end{cases}$$

$$H^{-n+1}(\mathrm{im}\{\mathrm{id} - \widetilde{T}_f\})_p \cong \mathrm{im}\{\mathrm{id} - \widetilde{T}^{-n+1}_{f,p}\},$$

$$H^{-n+1}(\mathrm{coker}\{\mathrm{id} - \widetilde{T}_f\})_p \cong \mathrm{coker}\{\mathrm{id} - \widetilde{T}^{-n+1}_{f,p}\},$$

where $\widetilde{T}^{-n+1}_{f,p}$ is the Milnor monodromy action on $H^1(F_{f,p};\mathbb{Q})$.

Proof. Since $H^k(\ker\{\mathrm{id} - \widetilde{T}_f\})_p = 0$ for $k \neq -n+1$, the result follows from the short exact sequences

$$0 \to H^{-n+1}(\ker\{\mathrm{id} - \widetilde{T}_f\})_p \to H^1(F_{f,p};\mathbb{Q}) \to H^{-n+1}(\mathrm{im}\{\mathrm{id} - \widetilde{T}_f\})_p \to 0,$$

and

$$0 \to H^{-n+1}(\mathrm{im}\{\mathrm{id} - \widetilde{T}_f\})_p \to H^1(F_{f,p};\mathbb{Q}) \to H^{-n+1}(\mathrm{coker}\{\mathrm{id} - \widetilde{T}_f\})_p \to 0.$$

$\square$

By taking stalk cohomology of the fundamental short exact sequence, we have

$$0 \to H^{-n}(\mathbb{Q}^\bullet_X[n])_p \to H^{-n}(\mathbf{I}^\bullet_X)_p \to \ker\{\mathrm{id} - \widetilde{T}^{-n+1}_{f,p}\} \to 0.$$

Since $\pi_*\mathbf{I}^\bullet_{\widetilde{X}} \cong \mathbf{I}^\bullet_X$, and $H^{-n}(\pi_*\mathbf{I}^\bullet_{\widetilde{X}})_p \cong \mathbb{Q}^{|\pi^{-1}(p)|}$,

$$\ker\{\mathrm{id} - \widetilde{T}^{-n+1}_{f,p}\} \cong \mathbb{Q}^{|\pi^{-1}(p)|-1}$$

for all $p \in X$, yielding the following nice lower-bound:

Corollary 1.2.0.2.

$$\dim_{\mathbb{Q}} H^1(F_{f,p};\mathbb{Q}) \geq |\pi^{-1}(p)| - 1.$$

1.3 Milnor Fibers in Parameterized Spaces

Throughout this section, we will assume that X is a parameterized space (see Definition 1.1.1.7). All results in this section hold with $\mathbb{Z}$ coefficients for smooth normalizations, and with $\mathbb{Q}$ coefficients for $\mathbb{Q}$-homology manifold normalizations. **Throughout this Section and Chapter 2, we will use the words "normalization" and "parameterization" interchangeably.** We will fix our base ring as $\mathbb{Z}$ unless explicitly stated otherwise. The results of this section are largely from the paper [23] by the author and David Massey.

1.3.1 Functions with Arbitrary Singularities

Let $h : (X, \mathbf{0}) \to (\mathbb{C}, 0)$ be a complex analytic function. We are interested in results on the Milnor fiber, $F_{h,\mathbf{0}}$ of h at $\mathbf{0}$. We remind the reader that, in this context in which the domain of h is allowed to be singular, a Milnor fibration still exists by the result of Lê in [28], and the Milnor fiber at a point $x \in V(h)$, is given by

$$F_{h,x} \;=\; B_\epsilon(x) \cap X \cap h^{-1}(a),$$

where $B_\epsilon(x)$ is an open ball of radius ϵ, centered at x, and $0 < |a| \ll \epsilon \ll 1$ (and, technically, the intersection with X is redundant, but we wish to emphasize that this Milnor fiber lives in X). We also care about the real link, $K_{X,x}$, of X at $x \in X$ [55], which is given by

$$K_{X,x} \;:= S_\epsilon(x) \cap X,$$

where, again, $0 < \epsilon \ll 1$, and $S_\epsilon(x)$ is the $2N + 1$ sphere of radius ϵ centered at x.

We will need to consider the Milnor fiber of $h \circ \pi$ at each of the p_i and the Milnor fiber of h restricted to the X_k's, which are equal to the intersections $X_k \cap F_{h,\mathbf{0}}$.

As X itself may be singular, it is important for us to say what notion we will use for a "critical point" of h. We use the Milnor fiber to define:

Definition 1.3.1.1. *The **topological/cohomological critical locus of** h, is*

$$\begin{aligned}\Sigma_{\text{top}} h &:= \overline{\{x \in V(h) \mid F_{h,x} \text{ does not have the integral cohomology of a point}\}}\\ &= \operatorname{supp} \phi_h[-1]\mathbb{Z}^\bullet_X[n]\end{aligned}$$

Remark 1.3.1.2. Why is this the correct definition of critical locus? There are many possible choices (cf. [38]). Since we are primarily concerned with the local topology of the hypersurface $V(h)$, it is most natural to record points $p \in V(h)$ over which this local topology changes, following the strategy of (stratified) Morse theory.

Since X is parameterized, $\mathbb{Z}^\bullet_X[n]$ is perverse; as $\phi_h[-1]$ takes perverse sheaves to perverse sheaves, $\phi_h[-1]\mathbb{Z}^\bullet_X[n]$ is a perverse sheaf supported on $\Sigma_{\rm top}h$. Consequently, if $\dim_{\mathbf{0}} \Sigma_{\rm top}h = 0$, $\phi_h[-1]\mathbb{Z}^\bullet_X[n]$ is a perverse sheaf on a point, and therefore has stalk cohomology only in degree 0 at $\mathbf{0}$, where we have

$$H^0(\phi_h[-1]\mathbb{Z}^\bullet_X[n])_{\mathbf{0}} \cong \widetilde{H}^{n-1}(F_{h,\mathbf{0}};\mathbb{Z}).$$

Definition 1.3.1.3. *If* $\mathbf{0}$ *is an isolated point in* $\Sigma_{_{\rm top}}h$*, then we define the* ***Milnor number of*** h ***at*** $\mathbf{0}$*,* $\mu_{\mathbf{0}}(h)$*, to be the rank of* $\widetilde{H}^{n-1}(F_{h,\mathbf{0}};\mathbb{Z})$*.*

The following theorem is now easy to prove.

Theorem 1.3.1.4. *There is a long exact sequence, relating the Milnor fiber of* h*, the Milnor fibers of* $h\circ\pi$*, and the hypercohomology of the Milnor fiber of* h *restricted to* D *with coefficients in* $\mathbf{N}^\bullet_X$*, given by*

$$\cdots \to \mathbb{H}^{j-n+1}(D\cap F_{h,\mathbf{0}};\mathbf{N}^\bullet_X) \to \widetilde{H}^j(F_{h,\mathbf{0}};\mathbb{Z}) \to$$

$$\bigoplus_i \widetilde{H}^j(F_{h\circ\pi,p_i};\mathbb{Z}) \to \mathbb{H}^{j-n+2}(D\cap F_{h,\mathbf{0}};\mathbf{N}^\bullet_X) \to \cdots \ .$$

This long exact sequence is compatible with the Milnor monodromy automorphisms in each degree.

Proof. We apply the exact functor $\phi_h[-1]$ to the short exact sequence (1.2) which defines $\mathbf{N}^\bullet_X$ to obtain the following short exact sequence of perverse sheaves:

$$0 \to \phi_h[-1]\mathbf{N}^\bullet \to \phi_h[-1]\mathbb{Z}^\bullet_X[n] \xrightarrow{\hat{\Delta}} \phi_h[-1]\pi_*\mathbb{Z}^\bullet_{\widetilde{X}}[n] \to 0, \tag{1.9}$$

where $\hat{\Delta} = \phi_h[-1]\Delta$. As the Milnor monodromy automorphism is natural, the maps in this short exact sequence commute with the Milnor monodromies.

If we let $\widehat{\pi}$ denote the restriction of π to a map from $(h\circ\pi)^{-1}(0)$ to $h^{-1}(0)$, then there is the well-known natural base change isomorphism (see Exercise VIII.15 of [25] or Proposition 4.2.11 of [6]):

$$\phi_h[-1]\pi_*\mathbb{Z}^\bullet_{\widetilde{X}}[n] \cong \widehat{\pi}_*\phi_{h\circ\pi}[-1]\mathbb{Z}^\bullet_{\widetilde{X}}[n].$$

By the induced long exact sequence on stalk cohomology and the lemma, we are finished. $\square$

Corollary 1.3.1.5. *If $\pi^{-1}(\mathbf{0}) \cap \Sigma_{\text{top}}(h \circ \pi) = \emptyset$, then there is an isomorphism*

$$\widetilde{H}^j(F_{h,\mathbf{0}}; \mathbb{Z}) \cong \mathbb{H}^{j-n+1}(D \cap F_{h,\mathbf{0}}; \mathbf{N}^\bullet_X)$$

and this isomorphism commutes with the Milnor monodromies.

Example 1.3.1.6. Suppose that we have a finite map $\Psi_0 : (\mathcal{V}, S) \to (\Omega, \mathbf{0})$, where $\mathcal{V}$ and Ω are open neighborhoods of S in $\mathbb{C}^d$ and of the origin in $\mathbb{C}^{d+1}$, respectively. Suppose that $\mathcal{T}$ is an open neighborhood of the origin in $\mathbb{C}^d$, and that $\Psi : \mathcal{T} \times \mathcal{V} \to \mathcal{T} \times \Omega$ is an unfolding of $\Psi = \Psi_{\mathbf{0}}$, i.e., Ψ is a finite analytic map of the form $\Psi(\mathbf{t}, \mathbf{v}) = (\mathbf{t}, \Psi_{\mathbf{t}}(\mathbf{v}))$, where, for each $\mathbf{t} \in \mathcal{T}$, $\Psi_{\mathbf{t}}$ is a finite map from $\mathcal{V}$ to Ω.

Let X denote the image of Ψ, continue to write Ψ for the map from $\mathcal{T} \times \mathcal{V}$ to X, and let h be the projection onto the first coordinate; thus, $(h \circ \Psi)(t_1, \dots, t_d, \mathbf{v}) = t_1$. Then, $S \cap \Sigma(h \circ \Psi) = \emptyset$ and so $\widetilde{H}^j(F_{h,\mathbf{0}}; \mathbb{Z})$ is isomorphic to

$$\widetilde{\mathbb{H}}^{j-n+1}(D \cap F_{h,\mathbf{0}}; \mathbf{N}^\bullet_X)$$

by an isomorphism which commutes with the Milnor monodromies.

Before we can prove the next corollary, we need to recall a lemma, which is well-known to experts in the field. See, for instance, [6], Theorem 4.1.22 (note that the setting of [6], Theorem 4.1.22, is algebraic, but that assumption is used in the proof only to guarantee that there are a finite number of strata).

Lemma 1.3.1.7. *Let $\mathfrak{S}$ be a complex analytic Whitney stratification, with connected strata, of a complex analytic space Y. Suppose that $\mathfrak{S}$ contains a finite number of strata. Let $\mathbf{A}^\bullet$ be a bounded complex of $\mathbb{Z}$-modules which is constructible with respect to $\mathfrak{S}$. For each stratum S, let p_S denote a point in S.*

Then, there is the following additivity/multiplicativity formula for the Euler characteristics:

$$\chi(Y; \mathbf{A}^\bullet) = \sum_S \chi(S)\chi(\mathbf{A}^\bullet)_{p_S}.$$

Corollary 1.3.1.8. *The relationship between the reduced Euler characteristics of the Milnor fiber of h at $\mathbf{0}$, the Milnor fibers of $h \circ \pi$, and the X_k's is given by*

$$\widetilde{\chi}(F_{h,\mathbf{0}}) = |\pi^{-1}(\mathbf{0})| - 1 + \sum_i \widetilde{\chi}(F_{h\circ\pi,p_i}) - \sum_{k\geq 2}(k-1)\chi(X_k \cap F_{h,\mathbf{0}}).$$

Proof. Via additivity of the Euler characteristic in the hypercohomology long exact sequence given in Theorem 1.3.1.4, we obtain the following relation:

$$\begin{aligned}\widetilde{\chi}(F_{h,\mathbf{0}}) &= \sum_i \widetilde{\chi}\left(F_{h\circ\pi,p_i}\right) - (-1)^{-n+1}\chi\left(\mathbb{H}^*(D \cap F_{h,\mathbf{0}}; \mathbf{N}^\bullet_X)\right) \\ &= |\pi^{-1}(\mathbf{0})| - 1 + \sum_i \widetilde{\chi}\left(F_{h\circ\pi,p_i}\right) - (-1)^{-n+1}\chi\left(\mathbb{H}^*(D \cap F_{h,\mathbf{0}}; \mathbf{N}^\bullet_X)\right).\end{aligned}$$

We are then finished, provided that we show that

$$(-1)^{-n+1}\chi(D \cap F_{h,\mathbf{0}}; \mathbf{N}^\bullet_X) \ = \ \sum_{k\geq 2}(k-1)\chi(X_k \cap F_{h,\mathbf{0}}).$$

For this, we use Lemma 1.3.1.7. Take a complex analytic Whitney stratification $\mathfrak{S}'$ of D such that $\mathbf{N}^\bullet_{X|_D}$ is constructible with respect to $\mathfrak{S}'$; hence, for each k, $D \cap X_k$ is a union of strata. As $F_{h,\mathbf{0}}$ transversely intersects these strata, there is an induced Whitney stratification $\mathfrak{S} = \{S\}$ on $D \cap F_{h,\mathbf{0}}$ and also on each $D \cap X_k \cap F_{h,\mathbf{0}}$; these stratifications have a finite number of strata, since the Milnor fiber is defined inside a small ball and $\mathfrak{S}'$ is locally finite.

Now, since the Euler characteristic of the stalk cohomology of $\mathbf{N}^\bullet_X$ at a point $x \in X_k$ is $(-1)^{-n+1}(k-1)$, Lemma 1.3.1.7 yields

$$\chi(D \cap F_{h,\mathbf{0}}; \mathbf{N}^\bullet_X) = (-1)^{-n+1}\sum_k \sum_{S \subseteq D\cap X_k \cap F_{h,\mathbf{0}}} (k-1)\chi(S).$$

Finally, we "put back together" the Euler characteristics of the X_k's, i.e.,

$$\chi(X_k \cap F_{h,\mathbf{0}}) = \sum_{S \subseteq D\cap X_k \cap F_{h,\mathbf{0}}} \chi(S),$$

by again applying Lemma 1.3.1.7 to the constant sheaf on $X_k \cap F_{h,\mathbf{0}}$. □

Remark 1.3.1.9. We did not need to use $\mathbf{N}^\bullet_X$ to prove Corollary 1.3.1.8. It follows quickly from the base change isomorphism which appears in the proof of Theorem 1.3.1.4, but, having the theorem, it seems natural to use it in the proof.

1.3.2 Functions with Isolated Singularities

The case where the function $h : (X, \mathbf{0}) \to (\mathbb{C}, 0)$ has $\mathbf{0}$ as an isolated point in $\Sigma_{_{\rm top}} h$ is of particular interest; indeed, if h is a generic linear form on $\mathcal{U}$, then $\dim_{\mathbf{0}} \Sigma_{\rm top} h \leq 0$ and $F_{h,\mathbf{0}}$ represents the **complex link** of X at $\mathbf{0}$.

Theorem 1.3.2.1. *Suppose that $\mathbf{0}$ is an isolated point in $\Sigma_{_{\rm top}} h$. Then,*

1. *for all $p_i \in \pi^{-1}(\mathbf{0})$, $\dim_{p_i} \Sigma(h \circ \pi) \leq 0$,*

2. *$\mathbb{H}^k(D \cap F_{h,\mathbf{0}}; \mathbf{N}^\bullet_X)$ is non-zero in (at most) in degree $k = 0$, where it is free Abelian, and*

3. *the reduced, integral cohomology of $F_{h,\mathbf{0}}$ is non-zero in, at most, one degree, degree $n-1$, where it is free Abelian of rank*

$$\begin{aligned}\mu_{\mathbf{0}}(h) &= \Big[\sum_i \mu_{p_i}(h \circ \pi)\Big] + \operatorname{rank} \mathbb{H}^0(D \cap F_{h,\mathbf{0}}; \mathbf{N}^\bullet_X) \\ &= \Big[\sum_i \mu_{p_i}(h \circ \pi)\Big] + (-1)^{n-1}\Big[(|\pi^{-1}(\mathbf{0})| - 1) - \sum_{k \geq 2}(k-1)\chi(X_k \cap F_{h,\mathbf{0}})\Big].\end{aligned}$$

4. *In particular, if $\mathbf{0}$ is an isolated point in $\Sigma_{_{\rm top}} h$ and $\pi^{-1}(\mathbf{0}) \cap \Sigma(h \circ \pi) = \emptyset$, then*

$$\mu_{\mathbf{0}}(h) = \operatorname{rank} \mathbb{H}^0(D \cap F_{h,\mathbf{0}}; \mathbf{N}^\bullet_X) = (-1)^{n-1}\Big[(|\pi^{-1}(\mathbf{0})| - 1) - \sum_{k \geq 2}(k-1)\chi(X_k \cap F_{h,\mathbf{0}})\Big].$$

Proof. Except for the last equalities in each line, this follows from the fact that $\phi_h[-1]$ is perverse exact and supported on the topological critical locus of h, and the short exact sequence (1.9) in the proof of Theorem 1.3.1.4, since the hypothesis is equivalent to $\mathbf{0}$ being an isolated point in the support of $\phi_h[-1]\mathbb{Z}_X[n]$, and perverse sheaves which are supported at just an isolated point have non-zero stalk cohomology in only one degree, namely degree 0.

The final equalities in each line follow from Corollary 1.3.1.8. $\square$

Remark 1.3.2.2. Let us return to the unfolding situation in Example 1.3.1.6, but now suppose that the normalization π of X is a stable unfolding of π_0 with an isolated instability (see Appendix B). Then, as before, letting h be a projection onto an unfolding coordinate, $\mathbf{0}$ is an isolated point in $\Sigma_{_{\text{top}}}h$ and $\pi^{-1}(\mathbf{0}) \cap \Sigma(h \circ \pi) = \emptyset$.

Thus, the stable fiber has the cohomology of a finite bouquet of $(n-1)$-spheres, where the number of spheres, the Milnor number, is given by

$$\operatorname{rank} \mathbb{H}^0(D \cap F_{h,\mathbf{0}}; \mathbf{N}^\bullet_X) = (-1)^{n-1}\Big[(|\pi^{-1}(\mathbf{0})| - 1) - \sum_{k \geq 2}(k-1)\chi(X_k \cap F_{h,\mathbf{0}})\Big].$$

Note, in particular, that this implies that the right-hand side is non-negative, which is distinctly non-obvious.

Example 1.3.2.3. Consider the simple, but illustrative, specific example where $\pi_0(u) = (u^2, u^3)$, and the stable unfolding is given by $\pi(t, u) = (t, u^2 - t, u(u^2 - t))$ (and $|\pi^{-1}(\mathbf{0})| = 1$). Let X be the image of π, and let $h : X \to \mathbb{C}$ be the projection onto the first coordinate, so that $(h \circ \pi)(t, u) = t$. Note that, using (t, x, y) as coordinates on $\mathbb{C}^3$, we have $X = V(y^2 - x^3 - tx^2)$.

Clearly $\mathbf{0} \notin \Sigma(h \circ \pi)$, and $\mathbf{0}$ is an isolated point in $\Sigma_{_{\text{top}}}h$. For $k \geq 2$, the only X_k which is not empty is X_2, which equals the t-axis minus the origin. Furthermore, $X_2 \cap F_{h,\mathbf{0}}$ is a single point.

We conclude from Theorem 1.3.2.1 that $F_{h,\mathbf{0}}$, which is the complex link of X, has the cohomology of a single 1-sphere.

As a further application, we recover a classical formula for the Milnor number, as given in Theorem 10.5 of [55]:

Theorem 1.3.2.4 (Milnor)**.** *Suppose that $n = 2$ and that π is a one-parameter unfolding of a parameterization π_0 of a plane curve singularity $X_0 = V(f_0)$ with r irreducible components at the origin. Let t be the unfolding parameter and suppose that the only singularities of*

$F_{t_{|X},\mathbf{0}}$ are nodes, and that there are δ of them. Then, $X = V(f)$ for some f with $f_0 := f_{|V(t)}$, and the Milnor number of f_0 is given by the formula:

$$\mu_{\mathbf{0}}\left(f_0\right) = 2\delta - r + 1.$$

Proof. We recall the following formula for the Milnor number of $f_{|V(t)}$ at $\mathbf{0}$ [42]:

$$\mu_{\mathbf{0}}\left(f_{|V(t)}\right) = \left(\Gamma^1_{f,t} \cdot V(t)\right)_{\mathbf{0}} + \left(\Lambda^1_{f,t} \cdot V(t)\right)_{\mathbf{0}},$$

where $\Gamma^1_{f,t}$ is the relative polar curve of g with respect to t, and $\Lambda^1_{g,t}$ is the one-dimensional Lê cycle of f with respect to t (see Appendix A).

Using that the only singularities of $F_{t_{|X},\mathbf{0}}$ are nodes, we immediately have $\left(\Lambda^1_{f,t} \cdot V(t)\right)_{\mathbf{0}} = \delta$. Since the unfolding function π has an isolated instability at $\mathbf{0}$, $\mu_{\mathbf{0}}(t_{|X})$ is equal to $\left(\Gamma^1_{f,t} \cdot V(t)\right)_{\mathbf{0}}$ (see, for example, [35]).

Now, Corollary 1.3.1.8 tells us that

$$\mu_{\mathbf{0}}(t_{|X}) = -r + 1 + \sum_{k \geq 2}(k-1)\chi(X_k \cap F_{t_{|X},\mathbf{0}}),$$

since $r = |\pi_0^{-1}(\mathbf{0})|$. By assumption, $\chi(X_2 \cap F_{t_{|X},\mathbf{0}})$ is the only non-zero summand in the above equation, and it is immediately seen to be the number of double points of $X \cap V(t)$ appearing in a stable perturbation. Thus,

$$\mu_{\mathbf{0}}\left(f_0\right) = 2\delta - r + 1$$

as desired. □

Chapter 2

Generalizing Milnor's Formula to Higher Dimensions

In re-proving Milnor's formula (Theorem 1.3.2.4), one immediately notices that the generality of the methods used are not at all limited to deformations of curves in $\mathbb{C}^2$; consequently, it is natural to hope that a similar, more general result holds between the vanishing cycles and $\mathbf{N}^\bullet_X$ in deformations of parameterized hypersurfaces. We prove such a generalization in this section, and obtain a similar formula for deformations of parameterized surfaces in $\mathbb{C}^3$, and a "bootstrap ansatz" for obtaining such results for deformations of parameterized hypersurfaces in $\mathbb{C}^{n+1}$ if one knows all of the stable maps from $\mathbb{C}^{n+1}$ to $\mathbb{C}^{n+2}$.

The first question we ask is: **what if we didn't have such a "stable" deformation of the curve $V(f_0)$?** That is, what if we didn't know that the origin $\mathbf{0} \in V(f_0)$ splits into δ nodes? We can still use the techniques of Theorem 1.3.2.4 of [23] in this situation. In this case, if π parameterizes the deformation of $V(f_0)$, we have

$$\mu_{\mathbf{0}}(f_0) = -m(\mathbf{0}) + \sum_{p \in B_\epsilon \cap V(t-t_0)} (\mu_p(f_{t_0}) + m(p)) \tag{2.1}$$

where $m(p) := |\pi^{-1}(p)| - 1$ (see Formula 1.4); the above formula follows easily from the same proof as Theorem 1.3.2.4.

Suppose now that $\pi_0 : (\widetilde{V(f_0)}, S) \to (V(f_0), \mathbf{0})$ is the normalization of a (reduced) hypersurface $V(f_0) \subseteq \mathbb{C}^n$, and π is a one-parameter unfolding of π_0 (see Section 2.2), so that, if $\mathbb{D}$ is a small open disk around the origin in $\mathbb{C}$, $\pi : (\mathbb{D} \times \widetilde{V(f_0)}, \{0\} \times S) \to (V(f), \mathbf{0})$ for some complex analytic function $f \in \mathcal{O}_{\mathbb{C}^{n+1},\mathbf{0}}$, where π is of the form $\pi(t, \mathbf{z}) = (t, \pi_t(\mathbf{z}))$ and $\pi(0, \mathbf{z}) = \pi_0(\mathbf{z})$.Here, $S = \pi_0^{-1}(\mathbf{0})$ is a finite subset of $\widetilde{V(f_0)}$, a purely $(n-1)$-dimensional $\mathbb{Q}$-homology (or smooth) manifold.

What would it mean to have a generalization of Formula 2.1? In the broadest sense, one would want to express numerical data about the singularities of f_0 completely in terms of data about the singularities of f_{t_0}, for t_0 small and non-zero. What changes when we move to higher dimensions?

One of the restrictions in considering parameterizable hypersurfaces $V(f)$ is that they must have codimension-one singularities. In particular, to get the most use out of the complex $\mathbf{N}^\bullet_{V(f)}$ on $V(f)$, we will assume the image multiple-point set $D = \operatorname{supp} \mathbf{N}^\bullet_{V(f)} \neq \emptyset$ and $D = \Sigma f$. For parameterized spaces, one always has the inclusion $D \subseteq \Sigma f$, but it is possible for this inclusion to be strict (e.g., if one parameterizes the cusp $y^2 = x^3$ in $\mathbb{C}^2$, or more generally, if $V(f)$ itself is a $\mathbb{Q}$-homology manifold). Since D is purely $(n-1)$-dimensional, we are stuck with hypersurfaces that have codimension-one singularities.

Consequently, we may no longer use the Milnor number in higher dimensions, since this number applies only to isolated singularities. One natural generalization of the Milnor number to higher-dimensional singularities are the **Lê numbers** $\lambda^i_{f,\mathbf{z}}$, and we will express the Lê numbers of the $t = 0$ slice of in terms of the Lê numbers of the $t \neq 0$ slice, together with the **characteristic polar multiplicities** of $\mathbf{N}^\bullet_{V(f)}$, which generalize the rank of the hypercohomology group $\mathbb{H}^0(D \cap F_{t_{|V(f)},\mathbf{0}}; \mathbf{N}^\bullet_{V(f)})$ used in Theorem 1.3.2.1. This will be explored in Section 2.2 and Section 2.3.

When moving to higher dimensions, we must also consider which sort of deformation to allow when relating f_0 and f_{t_0} for t_0 small and not zero. For this, we choose the notion of a deformation with isolated polar activity (or, an **IPA-deformation**). Intuitively, these are deformations where the only "interesting" behavior happens at the origin, and the only change propagates outwards from the origin along curves. Such deformations exist generically in all dimensions. We examine this notion, first introduced by Massey in [41], in Section 2.1. An ordered tuple of linear forms $\mathbf{z} = (z_0, \cdots, z_k)$ is called an IPA-tuple (for f at $\mathbf{0}$) if, for $1 \leq i \leq k$, $f_{|_{V(z_0,\cdots,z_{i-1})}}$ is an IPA-deformation of $f_{|_{V(z_0,\cdots,z_i)}}$ at $\mathbf{0}$.

In Section 2.4, we prove the following result.

Theorem 2.0.0.1 (Theorem 2.4.0.2)**.** *Suppose that* $\pi : (\mathbb{D} \times \widetilde{V(f_0)}, \{0\} \times S) \to (V(f), \mathbf{0})$ *is a one-parameter unfolding of a parameterized hypersurface* $\operatorname{im} \pi_0 = V(f_0)$*. Suppose further that* $\mathbf{z} = (z_1, \cdots, z_n)$ *is chosen such that* $\mathbf{z}$ *is an IPA-tuple for* $f_0 = f_{|_{V(t)}}$ *at* $\mathbf{0}$*. Then, the following formulas hold for the Lê numbers of* f_0 *with respect to* $\mathbf{z}$ *at* $\mathbf{0}$*: for* $0 < |t_0| \ll \epsilon \ll 1$*,*

$$\lambda^0_{f_0,\mathbf{z}}(\mathbf{0}) = -\lambda^0_{\mathbf{N}^\bullet_{V(f_0)},\mathbf{z}}(\mathbf{0}) + \sum_{p \in B_\epsilon \cap V(t-t_0)} \left(\lambda^0_{f_{t_0},\mathbf{z}}(p) + \lambda^0_{\mathbf{N}^\bullet_{V(f_{t_0})},\mathbf{z}}(p) \right),$$

and, for $1 \leq i \leq n-2$*,*

$$\lambda^i_{f_0,\mathbf{z}}(\mathbf{0}) = \sum_{q \in B_\epsilon \cap V(t-t_0, z_1, z_2, \cdots, z_i)} \lambda^i_{f_{t_0},\mathbf{z}}(q).$$

In particular, the following relationship holds for $0 \leq i \leq n-2$*:*

$$\lambda^i_{f_0,\mathbf{z}}(\mathbf{0}) + \lambda^i_{\mathbf{N}^\bullet_{V(f_0)},\mathbf{z}}(\mathbf{0}) = \sum_{p \in B_\epsilon \cap V(t-t_0, z_1, z_2, \cdots, z_i)} \left(\lambda^i_{f_{t_0},\mathbf{z}}(p) + \lambda^i_{\mathbf{N}^\bullet_{V(f_{t_0})},\mathbf{z}}(p) \right).$$

We then conclude the chapter with some applications of this theorem to various dimensions, and obtain formulas in the same vein as Milnor's double point formula.

2.1 IPA-Deformations

Although we need to consider only the case of a family of parameterized hypersurfaces for this section, much of the machinery we use for Section 2.3 and Section 2.4 does not require such restrictive hypotheses. That is, the notion of IPA-deformations and Lê numbers (see

Massey, [41] and [42]) apply to hypersurface singularities in general, not just parameterized hypersurfaces.

Suppose $\mathbf{z} = (z_0, \cdots, z_n)$ are local coordinates on an open neighborhood $\mathcal{U} \subseteq \mathbb{C}^{n+1}$ of $\mathbf{0}$, so that we have $T^*\mathcal{U} \cong \mathcal{U} \times \mathbb{C}^{n+1}$, with fiber-wise basis $(d_p z_0, \cdots, d_p z_n)$ of $(T^*\mathcal{U})_p = \tau^{-1}(p)$, where $\tau : T^*\mathcal{U} \to \mathcal{U}$ is the canonical projection map.

Denote by $\mathrm{Span}\langle dz_0, \cdots, dz_k\rangle$ the subset of $T^*\mathcal{U}$ given by $\{(p, \sum_{i=0}^k w_i d_p z_i) \,|p \in \mathcal{U}, w_i \in \mathbb{C}\}$

Let $f : (\mathcal{U}, \mathbf{0}) \to (\mathbb{C}, 0)$ be a (reduced) complex analytic function, where $\mathcal{U}$ is a connected open neighborhood of the origin in $\mathbb{C}^{n+1}$.

Finally, let $\overline{T^*_f\mathcal{U}}$ denote the (closure of) the relative conormal space of f in $\mathcal{U}$, i.e.,

$$\overline{T^*_f\mathcal{U}} := \overline{\{(p, \xi) \in T^*\mathcal{U} \,|\, \xi(\ker d_p f) = 0\}}.$$

It is important to note that $\overline{T^*_f\mathcal{U}}$ is a $\mathbb{C}$-conic subset of $T^*\mathcal{U}$, as we will consider its projectivization in Definition 2.1.0.2.

The following definitions of the relative polar varieties of f differ slightly from their more classical construction (see, for example [19], [29], or [27]), following that of [41],[44]. Lastly, the intersection product appearing in the following definitions is that of proper intersections in complex manifolds (See Chapter 6 of [9]).

Definition 2.1.0.1. *The* ***relative polar curve of*** f ***with respect to*** z_0*, denoted* Γ^1_{f,z_0}*, is, as an analytic cycle at the origin, the collection of those components of the cycle*

$$\tau_*\left(\overline{T^*_f\mathcal{U}} \cdot \mathrm{im}\; dz_0\right)$$

which are not contained in Σf*, provided that* $\overline{T^*_f\mathcal{U}}$ *and* $\mathrm{im}\; dz_0$ *intersect properly in* $T^*\mathcal{U}$ *(where* τ_* *is the proper pushfoward of cycles).*

More generally, one can define the higher k-dimensional relative polar varieties $\Gamma^k_{f,\mathbf{z}}$ in this manner, by considering the *projectivized* relative conormal space $\mathbb{P}(\overline{T^*_f\mathcal{U}})$ as follows. For $0 \leq$

$k \leq n$, consider the subspace $\mathbb{P}(\text{Span}\langle dz_0, \cdots, dz_k\rangle)$ of $\mathbb{P}(T^*\mathcal{U}) \cong \mathcal{U} \times \mathbb{P}^n$, the projectivized cotangent bundle of $\mathcal{U}$ (The following definition **does not** require one to use the projectivized relative conormal space; we do so to make the formulas involved less cumbersome).

Definition 2.1.0.2. *The* $(k+1)$***-dimensional relative polar variety of*** f ***with respect to*** $\mathbf{z}$, *denoted* $\Gamma^k_{f,\mathbf{z}}$, *is, as an analytic cycle at the origin, the collection of those components of*

$$\tau_* \left(\mathbb{P}(\overline{T^*_f\mathcal{U}}) \cdot \mathbb{P}\left(\text{Span}\langle dz_0, \cdots, dz_k\rangle\right)\right)$$

which are not contained in the critical locus Σf *at the origin, provided that* $\mathbb{P}(\overline{T^*_f\mathcal{U}})$ *and* $\mathbb{P}\left(\text{Span}\langle dz_0, \cdots, dz_k\rangle\right)$ *intersect properly in* $T^*\mathcal{U}$*. By abuse of notation, we also use* τ *to denote the canonical projection* $\mathbb{P}(T^*\mathcal{U}) \to \mathcal{U}$*.*

See Appendix A for the classical definition of $\Gamma^k_{f,\mathbf{z}}$.

Throughout this section (and, this thesis in general), we will use the (shifted) **nearby and vanishing cycle functors** $\psi_f[-1]$ and $\phi_f[-1]$, respectively, from the bounded derived category $D^b_c(\mathcal{U})$ of constructible complexes of sheaves on $\mathcal{U}$ to those on $V(f)$ (see for example [25], [6], [18], or [2], or Section 1.3). The shifts $[-1]$ are need to for these functors to take perverse sheaves on $\mathcal{U}$ to perverse sheaves on $V(f)$. One of the most important properties of these functors is that, for an arbitrary bounded, constructible complex of sheaves $\mathbf{F}^\bullet$ on $\mathcal{U}$, we have isomorphisms

$$H^k(\psi_f[-1]\mathbf{F}^\bullet)_p \cong \mathbb{H}^k(F_{f,p}; \mathbf{F}^\bullet) \text{ and} \tag{2.2}$$

$$H^k(\phi_f[-1]\mathbf{F}^\bullet)_p \cong \mathbb{H}^{k+1}(B_\epsilon(p), F_{f,p}; \mathbf{F}^\bullet), \tag{2.3}$$

where $\mathbb{H}^*$ denotes hypercohomology of complexes of sheaves, and $F_{f,p} = B_\epsilon(p) \cap f^{-1}(\xi)$ denotes the Milnor fiber of f at p (here $0 < |\xi| \ll \epsilon \ll 1$). If we use $\mathbb{Z}^\bullet_\mathcal{U}[n+1]$ for coefficients, then $\psi_f[-1]$ (resp. $\phi_f[-1]$) recovers the ordinary integral (resp. reduced) cohomology groups of the Milnor fiber $F_{f,p}$ of f at p (up to a shift):

$$H^k(\phi_f[-1]\mathbb{Z}^\bullet_\mathcal{U}[n+1])_p \cong \widetilde{H}^{k+n}(F_{f,p}; \mathbb{Z}).$$

We will also make frequent use of the **microsupport** $SS(\mathbf{F}^\bullet)$ of a (bounded, constructible) complex of sheaves $\mathbf{F}^\bullet$ which is a closed $\mathbb{C}^\times$-conic subset of $T^*\mathcal{U}$. We will use the following characterization of $SS(\mathbf{F}^\bullet)$ in terms the vanishing cycles (See Prop 8.6.4, of [25]).

Proposition 2.1.0.3 (Microsupport)**.** *Let $\mathbf{F}^\bullet \in D_c^b(\mathcal{U})$ and let $(p,\xi) \in T^*\mathcal{U}$. Then, the following are equivalent:*

1. $(p,\xi) \notin SS(\mathbf{F}^\bullet)$.
2. *There exists an open neighborhood Ω of (p,ξ) in $T^*\mathcal{U}$ such that, for any $q \in \mathcal{U}$ and any complex analytic function g defined in a neighborhood of q with $f(q) = 0$ and $(q, d_q g) \in \Omega$, one has $(\phi_g \mathbf{F}^\bullet)_q = 0$.*

It is instructive to think about the condition $(p, d_p g) \notin SS(\mathbf{F}^\bullet)$ from the perspective of microlocal/stratified Morse theory. That is, $(p, d_p g) \notin SS(\mathbf{F}^\bullet)$ if and only if p is not a critical point of g "with coefficients in $\mathbf{F}^\bullet$" (Definition 1.3.1.1).

In order to compute numerical invariants associated to certain perverse sheaves (see the characteristic polar multiplicities (Section 2.3) and Lê numbers), we need to choose linear forms that "cut down" the support in a certain way. We now give several equivalent conditions for this "cutting" procedure, that will be used throughout this paper (see Definition 2.1.0.6).

Proposition 2.1.0.4 (IPA-Deformations)**.** *The following are equivalent:*

1. $\dim_{\mathbf{0}} \Gamma^1_{f,z_0} \cap V(z_0) \leq 0$.
2. $\dim_{\mathbf{0}} \Gamma^1_{f,z_0} \cap V(f) \leq 0$.
3. $\dim_{(\mathbf{0}, d_{\mathbf{0}} z_0)} \operatorname{im} dz_0 \cap (f \circ \tau)^{-1}(0) \cap \overline{T^*_f \mathcal{U}} \leq 0$, *where again $\tau : T^*\mathcal{U} \to \mathcal{U}$ is the canonical projection map.*

4. $\dim_{(\mathbf{0},d_{\mathbf{0}}z_0)} SS(\psi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]) \cap \operatorname{im} dz_0 \leq 0$.

5. $\dim_{(\mathbf{0},d_{\mathbf{0}}z_0)} SS(\mathbb{Z}^\bullet_{V(f)}[n]) \cap \operatorname{im} dz_0 \leq 0$.

6. $\dim_{\mathbf{0}} \operatorname{supp} \phi_{z_0}[-1]\mathbb{Z}^\bullet_{V(f)}[n] \leq 0$.

7. *Away from* $\mathbf{0}$*, the comparison morphism* $\mathbb{Z}^\bullet_{V(f,z_0)}[n-1] \to \psi_{z_0}[-1]\mathbb{Z}^\bullet_{V(f)}[n]$ *is an isomorphism.*

Proof. The equivalence of statements (1), (2), and (3) are covered in Proposition 2.6 of [41].

The equivalence (3) $\iff$ (4) follows directly from the equality

$$\overline{T^*_f\mathcal{U}} \cap (f \circ \tau)^{-1}(0) = SS(\psi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]).$$

(See [5] for the original result, although the phrasing used above is found in [47]).

To see the equivalence (4) $\iff$ (5), consider the natural distinguished triangle

$$i_*i^*[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1] \to j_!j^!\mathbb{Z}^\bullet_{\mathcal{U}}[n+1] \to \mathbb{Z}^\bullet_{\mathcal{U}}[n+1] \xrightarrow{+1} \quad (\ddagger)$$

where $i : V(f) \hookrightarrow \mathcal{U}$, and $j : \mathcal{U}\backslash V(f) \hookrightarrow \mathcal{U}$. Then, by [51], there is an equality of microsupports

$$SS(\psi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]) = SS(j_!j^!\mathbb{Z}^\bullet_{\mathcal{U}}[n+1])_{\subseteq V(f)},$$

where the subscript $\subseteq V(f)$ denotes the union of irreducible components of $SS(j_!j^!\mathbb{Z}_{\mathcal{U}}[n+1])$ that lie over the hypersurface $V(f)$. But, since $SS(\mathbb{Z}_{\mathcal{U}}[n+1]) \cong \mathcal{U} \times \{\mathbf{0}\}$, $(\ddagger)$ implies that

$$SS(i_*i^*[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]) = SS(j_!j^!\mathbb{Z}^\bullet_{\mathcal{U}}[n+1])_{\subseteq V(f)},$$

by the triangle inequality for microsupports. The claim follows after noting $i_*i^*[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1] = \mathbb{Z}^\bullet_{V(f)}[n]$.

The equivalence (5) $\iff$ (6) follows easily from Proposition 2.1.0.3, or see Theorem 3.1 of [38].

Lastly, one concludes (6) $\iff$ (7) trivially from the short exact sequence of perverse sheaves

$$0 \to \mathbb{Z}^\bullet_{V(f,z_0)}[n-1] \to \psi_{z_0}[-1]\mathbb{Z}^\bullet_{V(f)}[n] \to \phi_{z_0}[-1]\mathbb{Z}^\bullet_{V(f)}[n] \to 0$$

on $V(f, z_0)$.

$\square$

Remark 2.1.0.5. [$\mathbb{Z}$ vs. $\mathbb{Q}$ coefficients] As we mentioned in the introduction of this section, all results hold with either $\mathbb{Z}$ coefficients or $\mathbb{Q}$ coefficients (depending on whether the normalization of $V(f)$ is smooth, or a $\mathbb{Q}$-homology manifold). To see this for Proposition 2.1.0.4, suppose that $\dim \operatorname{supp} \phi_L[-1]\mathbb{Q}^\bullet_{V(f)}[n] \leq 0$ but $\dim \operatorname{supp} \phi_L[-1]\mathbb{Z}^\bullet_{V(f)}[n] > 0$. Then, at a generic point p of $\operatorname{supp} \phi_L[-1]\mathbb{Z}^\bullet_{V(f)}[n]$, the stalk cohomology of $\phi_L[-1]\mathbb{Z}^\bullet_{V(f)}[n]$ is a torsion $\mathbb{Z}$-module concentrated in a single degree. However, this cohomology must be free Abelian (see e.g, Lê's classical result about the cohomology of the Milnor fiber, or Proposition 1.2.3 of [23]) and is therefore zero. The reverse implication, from $\mathbb{Z}$ to $\mathbb{Q}$ coefficients, is trivial.

Thus, $\dim_{\mathbf{0}} \operatorname{supp} \phi_L[-1]\mathbb{Q}^\bullet_{V(f)}[n] \leq 0$ if and only if $\dim_{\mathbf{0}} \operatorname{supp} \phi_L[-1]\mathbb{Z}^\bullet_{V(f)}[n] \leq 0$.

Definition 2.1.0.6. *Given an analytic function $f : (\mathcal{U}, \mathbf{0}) \to (\mathbb{C}, 0)$ and a non-zero linear form $z_0 : (\mathcal{U}, 0) \to (\mathbb{C}, 0)$, we say that f is a deformation of $f_{|_{V(z_0)}}$ with* ***isolated polar activity*** *at* $\mathbf{0}$ *(or, an* ***IPA-deformation*** *for short) if the equivalent statements of Proposition 2.1.0.4 hold.*

Remark 2.1.0.7. IPA-deformations are closely related to the notion of **prepolar deformations** [46]; given a Thom a_f stratification $\mathfrak{S}$ of $V(f)$ and linear form L, we say f is a prepolar deformation of $f_{|_{V(L)}}$ if $V(L)$ transversally intersects all strata $S \in \mathfrak{S}\backslash\{\mathbf{0}\}$ in a neighborhood of the origin. We can alternatively phrase this as

$$\dim_{\mathbf{0}} \bigcup_{S \in \mathfrak{S}} \Sigma\left(L_{|_S}\right) \leq 0,$$

where the union $\bigcup_{S \in \mathfrak{S}} \Sigma\left(L_{|_S}\right) =: \Sigma_{\mathfrak{S}} L_{|_{V(f)}}$ is called the **stratified critical locus of** $L_{|_{V(f)}}$ **with respect to** $\mathfrak{S}$ (see Definition 1.3 of [38]) .

In particular, a prepolar deformation is defined with respect to a given a_f stratification $\mathfrak{S}$, whereas an IPA-deformation does not refer to any stratification. While one does always have the inclusion

$$\operatorname{supp} \phi_L[-1]\mathbb{Z}^\bullet_{V(f)}[n] =: \Sigma_{\text{top}}\left(L_{|_{V(f)}}\right) \subseteq \Sigma_{\mathfrak{S}}\left(L_{|_{V(f)}}\right)$$

(this follows by Stratified Morse Theory, or Remark 1.10 of [38], or Proposition 8.4.1 and Exercise 8.6.12 of [25])), it is an open question whether or not there exist IPA deformations that are not prepolar deformations.

We can iterate the notion of an IPA-deformation as follows.

Definition 2.1.0.8. *Let $k \geq 0$. A $(k+1)-$tuple $(z_0, \cdots, z_k)$ is said to be an IPA-tuple for f at $\mathbf{0}$ if, for all $1 \leq i \leq k$, $f_{|_{V(z_0,\cdots,z_{i-1})}}$ is an IPA-deformation of $f_{|_{V(z_0,\cdots,z_i)}}$ at $\mathbf{0}$.*

The following lemma follows from an inductive application of Theorem 1.1 of [37], and is crucial for our understanding of what IPA-deformation "looks like" in the cotangent bundle (cf. Proposition 2.1.0.4, item (2)).

Lemma 2.1.0.9 (Gaffney, Massey, [15])**.** *Let $k \geq 0$. Then, for all $p \in V(z_0, \cdots, z_{k-1})$ with $d_p z_k \notin \left(\overline{T^*_{f_{|_{V(z_0,\cdots,z_{k-1})}}} V(z_0, \cdots, z_{k-1})}\right)_p$, we have*

$$\left(\overline{T^*_f \mathcal{U}}\right)_p \cap \operatorname{Span}\langle d_p z_0, \cdots, d_p z_k\rangle = 0.$$

The main goal of this subsection is the following result. This result, originally from [42], is presented here with the "weaker" hypothesis of choosing an IPA-tuple, in lieu of a prepolar-tuple. For the definition of the Lê numbers of f with respect to a tuple of linear forms $\mathbf{z}$, see Appendix A.

Proposition 2.1.0.10 (Existence of Lê Numbers of a Slice)**.** *Suppose that $\mathbf{z} = (z_0, \cdots, z_n)$ is an IPA-tuple for f at $\mathbf{0}$, and use coordinates $\widetilde{\mathbf{z}} = (z_1, \cdots, z_n)$ for $V(z_0)$. Then, for*

$0 \leq i \leq \dim_{\mathbf{0}} \Sigma f$, *the Lê numbers* $\lambda^i_{f,\mathbf{z}}(\mathbf{0})$ *are defined, and the following equalities hold:*

$$\lambda^0_{f_{|_{V(z_0)}},\tilde{\mathbf{z}}}(\mathbf{0}) = \left(\Gamma^1_{f,z_0} \cdot V(z_0)\right)_{\mathbf{0}} + \lambda^1_{f,\mathbf{z}}(\mathbf{0})$$

$$\lambda^i_{f_{|_{V(z_0)}},\tilde{\mathbf{z}}}(\mathbf{0}) = \lambda^{i+1}_{f,\mathbf{z}}(\mathbf{0}),$$

for $1 \leq i \leq \dim_{\mathbf{0}} \Sigma f - 1$, *where* Γ^1_{f,z_0} *is the relative polar curve of* f *with respect to* z_0*.*

Proof. The proof follows Theorem 1.28 of [42], *mutatis mutandis* (changing prepolar to IPA).

Via the Chain Rule, it suffices to demonstrate that

$$\dim_{\mathbf{0}} \Gamma^{i+1}_{f,\mathbf{z}} \cap V(f) \cap V(z_0, \cdots, z_{i-1}) \leq 0,$$

since any analytic curve in $\Gamma^{i+1}_{f,\mathbf{z}} \cap V\left(\frac{\partial f}{\partial z_i}\right) \cap V(z_0, \cdots, z_{i-1})$ passing through $\mathbf{0}$ must be contained in $V(f)$, where $\Gamma^{i+1}_{f,\mathbf{z}}$ is the $(i+1)$-dimensional relative polar variety of f with respect to $\mathbf{z}$ (Definition 2.1.0.2).

Suppose that we had a sequence of points $p \in \Gamma^{i+1}_{f,\mathbf{z}} \cap V(f) \cap V(z_0, \cdots, z_{i-1})$ approaching $\mathbf{0}$. As each p is contained in $\Gamma^{i+1}_{f,\mathbf{z}}$, for each p we can find a sequence $p_k \to p$ with $p_k \notin \Sigma f$ satisfying $\langle d_{p_k} f \rangle \subseteq \operatorname{Span}\langle d_{p_k} z_0, \cdots, d_{p_k} z_{i-1}\rangle$ for each k. But then, by construction, we have found a nonzero element in the intersection $\left(\overline{T^*_f \mathcal{U}}\right)_p \cap \operatorname{Span}\langle d_p z_0 \cdots, d_p z_{i-1}\rangle$, contradicting Lemma 2.1.0.9.

□

2.2 Unfoldings and $\mathbf{N}^\bullet_{V(f)}$

As mentioned in the introduction of this section, we will be considering parameterized hypersurfaces that are the total space of a family of parameterized hypersurfaces. We make this precise with the following definition.

Definition 2.2.0.1. *A parameterization* $\pi : (\mathbb{D} \times \widetilde{V(f_0)}, \{0\} \times S) \to (V(f), \mathbf{0})$ *is said to be a* ***one-parameter unfolding*** *with unfolding parameter* t *if* π *is of the form*

$$\pi(t, \mathbf{z}) = (t, \pi_t(\mathbf{z}))$$

where $\pi_0(\mathbf{z}) := \pi(0, \mathbf{z})$ is a generically one-to-one parameterization of $V(f, t)$.

We say that a parameterization π_0 has an **isolated instability** at $\mathbf{0}$ with respect to an unfolding π of π_0 with parameter t if one has $\dim_{\mathbf{0}} \Sigma_{\text{top}} t_{|_{\operatorname{im} \pi}} \leq 0$.

The following proposition is one of our main motivations for using IPA-deformations: they naturally appear from one-parameter unfoldings with isolated instabilities.

Proposition 2.2.0.2. *Suppose $\pi : (\mathbb{D} \times \widetilde{V(f_0)}, \{0\} \times S) \to (V(f), \mathbf{0})$ is a 1-parameter unfolding of π_0 with unfolding parameter t, such that π_0 has an isolated instability at $\mathbf{0}$ with respect to π. Then, f is an IPA-deformation of $f_{|_{V(t)}}$ at $\mathbf{0}$.*

Proof. By definition, π_0 has an isolated instability at $\mathbf{0}$ with respect to the unfolding π with parameter t if

$$\dim_{\mathbf{0}} \Sigma_{\text{top}} \left(t_{|_{V(f)}} \right) \leq 0.$$

Following Definition 1.9 of [38],

$$\begin{aligned} \Sigma_{\text{top}} \left(t_{|_{V(f)}} \right) &= \overline{\{p \in V(f) \mid (p, d_p t) \in SS(\mathbb{Z}^\bullet_{V(f)}[n])\}} \\ &= \tau \left(SS(\mathbb{Z}^\bullet_{V(f)}[n]) \cap \operatorname{im} dt \right), \end{aligned}$$

where $\tau : T^*\mathcal{U} \to \mathcal{U}$ is the natural projection. This follows immediately from Proposition 2.1.0.3.

Consequently, if $\dim_{\mathbf{0}} \Sigma_{\text{top}} \left(t_{|_{V(f)}} \right) \leq 0$, it follows that $(\mathbf{0}, d_{\mathbf{0}} t)$ is an isolated point in the intersection $SS(\mathbb{Z}^\bullet_{V(f)}[n]) \cap \operatorname{im} dt$, and the the result follows by Proposition 2.1.0.4. $\square$

Remark 2.2.0.3. It is well-known that finitely-determined map germs π_0 have isolated instabilities with respect to a generic one-parameter unfolding ([53] pg. 241, and [14]). Consequently, generic one-parameter unfoldings of finitely-determined maps parameterizing hypersurfaces all give IPA-deformations.

Remark 2.2.0.4. If π is a one-parameter unfolding of a parameterization π_0, then for all t_0 small, it is easy to see that there is an isomorphism $\mathbf{N}^\bullet_{V(f)_{|_{V(t-t_0)}}}[-1] \cong \mathbf{N}^\bullet_{V(f_{t_0})}$, where $\pi_{t_0}(\mathbf{z}) = \pi(t_0, \mathbf{z})$.

Example 2.2.0.5. In the situation of Milnor's double-point formula, $\pi : (\mathbb{D}\times\mathbb{C}, \{0\}\times S) \to (\mathbb{C}^3, \mathbf{0})$ parameterizes the deformation of the curve $V(f_0)$ with r irreducible components at $\mathbf{0}$ into a curve $V(f_{t_0})$ with only double-point singularities. Hence, $\dim_{\mathbf{0}} V(f) = 2$, and the image multiple-point set D is purely 1-dimensional at $\mathbf{0}$.

Since π is a one-parameter unfolding with parameter t, we moreover have

$$\mathbf{N}^\bullet_{V(f)_{|_{V(t-t_0)}}}[-1] \cong \mathbf{N}^\bullet_{V(f_{t_0})},$$

where $\mathbf{N}^\bullet_{V(f_{t_0})}$ is the multiple-point complex of the parameterization $\pi_{t_0}(\mathbf{z})$. For all $t_0 \neq 0$ small, $\mathbf{N}^\bullet_{V(f_{t_0})}$ is supported on the set of double points of $V(f_{t_0})$, and at each such double-point p we have $\operatorname{rank} H^0(\mathbf{N}^\bullet_{V(f_0)})_p = |\pi^{-1}(p)| - 1 = 1$.

At $\mathbf{0} \in V(f_0)$, we have $\pi^{-1}(\mathbf{0}) = S$, and $|S| = r$ by assumption. Thus, $\operatorname{rank} H^0(\mathbf{N}^\bullet_{V(f_0)})_{\mathbf{0}} = r - 1$.

2.3 Characteristic Polar Multiplicities

The central concept of this section, the characteristic polar multiplicities of a perverse sheaf, were first defined and explored in [45]. These multiplicities, defined with respect to a "nice" choice of a tuple of linear forms $\mathbf{z} = (z_0, \cdots, z_s)$, are non-negative, integer-valued functions that mimic the properties of the Lê numbers associated to non-isolated hypersurface singularities (see [42]), and the characteristic polar multiplicities of the vanishing cycles $\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]$ with respect to $\mathbf{z}$ coincide with the Lê numbers of f with respect to $\mathbf{z}$.

Definition 2.3.0.1 (Corollary 4.10 [45]). *Let $\mathbf{P}^\bullet$ be a perverse sheaf on $V(f)$, with $\dim_{\mathbf{0}} \operatorname{supp} \mathbf{P}^\bullet = s$. Let $\mathbf{z} = (z_0, \cdots, z_s)$ be a tuple of linear forms such that, for all $0 \leq i \leq s$, we have*

$$\dim_{\mathbf{0}} \operatorname{supp} \phi_{z_i - z_i(p)}[-1]\psi_{z_{i-1}-z_{i-1}(p)}[-1]\cdots\psi_{z_0-z_0(p)}[-1]\mathbf{P}^\bullet \leq 0.$$

Then, the i-dimensional ***characteristic polar multiplicity of*** $\mathbf{P}^\bullet$ *with respect to* $\mathbf{z}$ *at $p \in V(g)$ is defined and given by the formula*

$$\lambda^i_{\mathbf{P}^\bullet,\mathbf{z}}(p) = \operatorname{rank}_{\mathbb{Z}} H^0(\phi_{z_i-z_i(p)}[-1]\psi_{z_{i-1}-z_{i-1}(p)}[-1]\cdots\psi_{z_0-z_0(p)}\mathbf{P}^\bullet)_p.$$

Remark 2.3.0.2. In general, one can define the characteristic polar multiplicities of any object in the bounded, derived category of constructible sheaves on $V(f)$, but they are slightly more cumbersome to define, and no longer need to be non-negative.

Example 2.3.0.3. Let $f : \mathcal{U} \to \mathbb{C}$ be an analytic function, with $f(\mathbf{0}) = 0$, $\mathcal{U}$ an open neighborhood of the origin in $\mathbb{C}^{n+1}$, and $\dim_{\mathbf{0}} \Sigma f = s$. Then, $\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]$ is a perverse sheaf on $V(f)$, with support equal to $\Sigma f \cap V(f)$. Indeed, the containment $\operatorname{supp} \phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1] \subseteq \Sigma f \cap V(f)$ follows from the complex analytic Implicit Function Theorem. For the reverse containment, if $p \notin \operatorname{supp} \phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]$, then the Milnor monodromy on the nearby cycles is the identity morphism, so that the Lefschetz number of the monodromy cannot be zero; by A'Campo's result [1], we therefore have $p \notin V(f) \cap \Sigma f$.

We then have

$$\lambda^i_{f,\mathbf{z}}(p) = \lambda^i_{\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1],\mathbf{z}}(p)$$

for all $0 \leq i \leq s$, and all p in an open neighborhood of $\mathbf{0}$ [45].

Remark 2.3.0.4. By Massey (Theorem 3.4 [43]), if $\dim_{\mathbf{0}} \Sigma f = s$, then there is a chain complex of **free** Abelian groups

$$0 \xrightarrow{\partial_{s+1}} \mathbb{Z}^{\lambda^s_{f,\mathbf{z}}(p)} \xrightarrow{\partial_s} \mathbb{Z}^{\lambda^{s-1}_{f,\mathbf{z}}(p)} \xrightarrow{\partial_{s-1}} \cdots \xrightarrow{\partial_2} \mathbb{Z}^{\lambda^1_{f,\mathbf{z}}(p)} \xrightarrow{\partial_1} \mathbb{Z}^{\lambda^0_{f,\mathbf{z}}(p)} \xrightarrow{\partial_0} 0$$

satisfying $\ker \partial_j / \operatorname{im} \partial_{i+1} \cong \widetilde{H}^{n-j}(F_{f,\mathbf{0}}; \mathbb{Z})$. Since this complex is free, tensoring this complex with $\mathbb{Q}$ will compute $\widetilde{H}^{n-j}(F_{f,\mathbf{0}}; \mathbb{Q})$. Hence, we can use either $\mathbb{Z}$ or $\mathbb{Q}$ coefficients in when characterizing the Lê numbers $\lambda^i_{f,\mathbf{z}}(p)$ in terms of the characteristic polar multiplicities of the vanishing cycles.

Example 2.3.0.5. If $\dim_{\mathbf{0}} \Sigma f = 0$, any non-zero linear form z_0 suffices for this construction, since $\psi_{z_0}[-1]\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1] = 0$. Then, the only non-zero Lê number of f is $\lambda^0_{f,z_0}(\mathbf{0})$, and we have

$$\begin{aligned}\lambda^0_{f,z_0}(\mathbf{0}) &= \operatorname{rank}_{\mathbb{Z}} H^0(\phi_{z_0}[-1]\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1])_{\mathbf{0}} \\ &= \operatorname{rank}_{\mathbb{Z}} H^0(\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1])_{\mathbf{0}} \\ &= \text{Milnor number of } f \text{ at } \mathbf{0}.\end{aligned}$$

Example 2.3.0.6. If $\dim_{\mathbf{0}} \Sigma f = 1$, we need z_0 such that $\dim_{\mathbf{0}} \Sigma\left(f_{|_{V(z_0)}}\right) = 0$, and any non-zero linear form suffices for z_1. Then the only non-zero Lê numbers of f with respect to $\mathbf{z} = (z_0, z_1)$ are $\lambda^0_{f,\mathbf{z}}(\mathbf{0})$ and $\lambda^1_{f,\mathbf{z}}(p)$ for $p \in \Sigma f$. At $\mathbf{0}$, we have

$$\lambda^1_{f,\mathbf{z}}(\mathbf{0}) = \sum_{C \subseteq \Sigma f \text{ irr.comp. at } \mathbf{0}} \mathring{\mu}_C \left(C \cdot V(z_0)\right)_{\mathbf{0}},$$

where $\mathring{\mu}_C$ denotes the generic transverse Milnor number of f along $C \backslash \{0\}$.

Remark 2.3.0.7. Analogous to the Lê numbers $\lambda^i_{f,\mathbf{z}}(p)$, the characteristic polar multiplicities of a perverse sheaf may be expressed as intersection numbers. That is, suppose we have a perverse sheaf $\mathbf{P}^\bullet$ and a tuple of linear forms $\mathbf{z}$ such that, for all $0 \leq i \leq \dim_{\mathbf{0}} \operatorname{supp} \mathbf{P}^\bullet$, the characteristic polar multiplicities $\lambda^i_{\mathbf{P}^\bullet,\mathbf{z}}(p)$ are defined for all p in a neighborhood $\mathcal{U}$ of $\mathbf{0}$. Then, there is a unique collection of non-negative analytic cycles $\Lambda^i_{\mathbf{P}^\bullet,\mathbf{z}}$ called the **characteristic polar cycles** of $\mathbf{P}^\bullet$ with respect to $\mathbf{z}$ satisfying, for all $p \in \mathcal{U}$,

$$\lambda^i_{\mathbf{P}^\bullet,\mathbf{z}}(p) = \left(\Lambda^i_{\mathbf{P}^\bullet,\mathbf{z}} \cdot V(z_0 - p_0, \cdots, \cdots, z_{i-1} - p_{i-1})\right)_p.$$

These cycles can also be thought of as being defined by the constructible function $\chi(\mathbf{P}^\bullet)_p$, so that

$$\chi(\mathbf{P}^\bullet)_p := \sum_i (-1)^i H^i(\mathbf{P}^\bullet)_p = \sum_i (-1)^i \lambda^i_{\mathbf{P}^\bullet,\mathbf{z}}(p).$$

Example 2.3.0.8. To illustrate this method of computing the characteristic polar multiplicities, we will compute $\lambda^0_{\mathbf{N}^\bullet_{V(f)},\mathbf{z}}(\mathbf{0})$ and $\lambda^1_{\mathbf{N}^\bullet_{V(f)},\mathbf{z}}(\mathbf{0})$ for a triple point singularity in $\mathbb{C}^3$, e.g., $V(f) = V(xyz)$. Clearly $V(f)$ is parameterized (the normalization π of $V(xyz)$ separates the three planes into a disjoint union in three copies of $\mathbb{C}^3$), and so $\mathbf{N}^\bullet_{V(f)}$ has stalk cohomology concentrated in degree -1, implying

$$\chi(\mathbf{N}^\bullet)_p = -|\pi^{-1}(p)| + 1.$$

Away from the origin, on the singular locus of $V(xyz)$, $\chi(\mathbf{N}^\bullet_{V(f)})_p$ has value -1 everywhere, and so we can identify the 1-dimensional characteristic polar cycle of $\mathbf{N}^\bullet_{V(f)}$ as the sum of the lines of intersection of these three planes, each weighted by 1. Thus, $\lambda^1_{\mathbf{N}^\bullet_{V(f)},\mathbf{z}}(\mathbf{0}) = 3$. Since $\chi(\mathbf{N}^\bullet_{V(f)})_\mathbf{0} = -2$, we find that $\lambda^0_{\mathbf{N}^\bullet_{V(f)},\mathbf{z}}(\mathbf{0}) = 1$, from the equality

$$-2 = \chi(\mathbf{N}^\bullet_{V(f)})_\mathbf{0} = \lambda^0_{\mathbf{N}^\bullet_{V(f)},\mathbf{z}}(\mathbf{0}) - \lambda^1_{\mathbf{N}^\bullet_{V(f)},\mathbf{z}}(\mathbf{0}) = \lambda^0_{\mathbf{N}^\bullet_{V(f)},\mathbf{z}}(\mathbf{0}) - 3.$$

Remark 2.3.0.9. We will need the representation of the characteristic polar multiplicities as intersection numbers in Section 2.4 when we will use the dynamic intersection property for proper intersections to understand $\lambda^i_{\mathbf{N}^\bullet_{V(f)},\mathbf{z}}(0)$. By this, we mean the equality

$$\left(\Lambda^i_{\mathbf{P}^\bullet,\mathbf{z}} \cdot V(z_0, z_1, \cdots, z_{i-1})\right)_\mathbf{0} = \sum_{p \in B_\epsilon \cap \Lambda^i_{\mathbf{P}^\bullet,\mathbf{z}} \cap V(z_0 - t)} \left(\Lambda^i_{\mathbf{P}^\bullet,\mathbf{z}} \cdot V(z_0 - t, z_1, \cdots, z_{i-1})\right)_p$$

for $0 < |t| \ll \epsilon \ll 1$ (see chapter 6 of [9]). Additionally, we will make use of the fact that characteristic polar multiplicities of perverse sheaves are additive on short exact sequences in Section 2.4. Precisely, if

$$0 \to \mathbf{A}^\bullet \to \mathbf{B}^\bullet \to \mathbf{C}^\bullet \to 0$$

is a short exact sequence of perverse sheaves, and if coordinates $\mathbf{z}$ are generic enough so that $\lambda^i_{\mathbf{B}^\bullet,\mathbf{z}}(p)$ is defined, then $\lambda^i_{\mathbf{A}^\bullet,\mathbf{z}}(p)$ and $\lambda^i_{\mathbf{C}^\bullet,\mathbf{z}}(p)$ are defined, and

$$\lambda^i_{\mathbf{B}^\bullet,\mathbf{z}}(p) = \lambda^i_{\mathbf{A}^\bullet,\mathbf{z}}(p) + \lambda^i_{\mathbf{C}^\bullet,\mathbf{z}}(p).$$

(See Proposition 3.3 of [45].)

Lemma 2.3.0.10. *If π is a one-parameter unfolding (with parameter t) of a parameterization of $V(f,t)$ with isolated instability at the origin, then the 0-dimensional characteristic polar multiplicity of $\mathbf{N}^\bullet_{V(f)}$ with respect to t is defined, and*

$$\lambda^0_{\mathbf{N}^\bullet_{V(f)},t}(\mathbf{0}) = \lambda^0_{\mathbb{Z}^\bullet_{V(f)}[n],t}(\mathbf{0}) = \left(\Gamma^1_{f,t} \cdot V(t)\right)_\mathbf{0}.$$

Proof. If f is an IPA-deformation of $f_{|_{V(t)}}$ at $\mathbf{0}$, then $\dim_{\mathbf{0}} \operatorname{supp} \phi_t[-1]\mathbb{Z}^\bullet_{V(f)}[n] \le 0$, by Proposition 2.1.0.4. By Definition 2.3.0.1, this is precisely what is needed to define $\lambda^0_{\mathbb{Z}^\bullet_{V(f)}[n],t}(\mathbf{0})$. Then, by a proper base-change, we have $\phi_t\pi_* \cong \hat{\pi}_*\phi_{t\circ\pi}$, where $\hat{\pi} : V(t \circ \pi) \to V(f,t)$ is the pullback of π via the inclusion $V(f,t) \hookrightarrow V(f)$. But, because π is a one-parameter unfolding, $t \circ \pi$ is a linear form on affine space and has no critical points; hence, $\phi_{t\circ\pi}\mathbb{Z}^\bullet_{\mathcal{U}} = 0$.

Consequently, it follows from the short exact sequence of perverse sheaves

$$0 \to \phi_t[-1]\mathbf{N}^\bullet_{V(f)} \to \phi_t[-1]\mathbb{Z}^\bullet_{V(f)}[n] \to \phi_t[-1]\pi_*\mathbb{Z}^\bullet_{\mathbb{D}\times\widetilde{X}}[n] \to 0$$

that there is an equality $\lambda^0_{\mathbf{N}^\bullet_{V(f)},t}(\mathbf{0}) = \lambda^0_{\mathbb{Z}^\bullet_{V(f)}[n],t}(\mathbf{0})$, since the characteristic polar multiplicities are additive on short exact sequences.

It is then a classical result by Lê, Hamm, Teissier, and Siersma that, for sufficiently generic t,

$$\lambda^0_{\mathbb{Z}^\bullet_{V(f)}[n],t}(\mathbf{0}) = \left(\Gamma^1_{f,t} \cdot V(t)\right)_{\mathbf{0}};$$

the result in the generality of IPA-deformations is found in [35]. The claim follows. □

Remark 2.3.0.11. The unfolding condition is not needed for the characteristic polar multiplicities of $\mathbf{N}^\bullet_{V(f)}$ to be defined, but it **is needed** for the vanishing $\lambda^0_{\pi_*\mathbb{Z}^\bullet_{\mathbb{D}\times\widetilde{V(f_0)}}[n],t}(\mathbf{0}) = 0$ which yields the equalities of Lemma 2.3.0.10.

Example 2.3.0.12. Let us compute $\lambda^0_{\mathbf{N}^\bullet_{V(f)},t}(\mathbf{0})$ in the case where $V(f)$ is the Whitney umbrella, with defining function $f(x,y,t) = y^2 - x^3 - tx^2$. Then, we can realize $V(f)$ as the total space of the one-parameter unfolding $\pi(t,u) = (u^2-t, u(u^2-t), t)$ with parameter t, and Lemma 2.3.0.10 tells us that $\lambda^0_{\mathbf{N}^\bullet_{V(f)},t}(\mathbf{0})$ is equal to the intersection multiplicity $\left(\Gamma^1_{f,t} \cdot V(t)\right)_{\mathbf{0}}$. A quick computation tells us that the relative polar curve $\Gamma^1_{f,t}$ is equal to $V(3x+2t, y)$, and thus transversely intersects $V(t)$ at $\mathbf{0}$. Hence,

$$\lambda^0_{\mathbf{N}^\bullet_{V(f)},t}(\mathbf{0}) = \left(\Gamma^1_{f,t} \cdot V(t)\right)_{\mathbf{0}} = 1.$$

The iterated IPA-condition implies the higher characteristic polar multiplicities of $\mathbf{N}^\bullet_{V(f)}$ exist as well.

Theorem 2.3.0.13. *Suppose that $(t, \mathbf{z}) = (t, z_1, \cdots, z_n)$ is an IPA-tuple for g at $\mathbf{0}$. Then, for $0 \leq i \leq n-1$, the characteristic polar multiplicities $\lambda^i_{\mathbf{N}^\bullet_{V(f)}, (t,\mathbf{z})}(\mathbf{0})$ with respect to $(t, \mathbf{z})$ are defined, and the following equalities hold:*

$$\lambda^0_{\mathbf{N}^\bullet_{V(f_0)}, \mathbf{z}}(\mathbf{0}) = \lambda^1_{\mathbf{N}^\bullet_{V(f)}, (t,\mathbf{z})}(\mathbf{0}) - \lambda^0_{\mathbf{N}^\bullet_{V(f)}, (t,\mathbf{z})}(\mathbf{0}),$$

and, for $1 \leq i \leq n-2$,

$$\lambda^i_{\mathbf{N}^\bullet_{V(f_0)}, \mathbf{z}}(\mathbf{0}) = \lambda^{i+1}_{\mathbf{N}^\bullet_{V(f)}, (t,\mathbf{z})}(\mathbf{0}).$$

Proof. That $\lambda^0_{\mathbf{N}^\bullet_{V(f)}, (t,\mathbf{z})}(\mathbf{0})$ is defined is precisely the inequality $\dim_{\mathbf{0}} \operatorname{supp} \phi_t[-1]\mathbf{N}^\bullet_{V(f)} \leq 0$ concluded in Lemma 2.3.0.10 from the short exact sequence

$$0 \to \phi_t[-1]\mathbf{N}^\bullet_{V(f)} \to \phi_t[-1]\mathbb{Z}^\bullet_{V(f)}[n] \to \phi_t[-1]\pi_*\mathbb{Z}^\bullet_{\mathbb{D} \times \widetilde{V(f_0)}}[n] \to 0.$$

By Proposition 3.2 of [45], it remains to show that $\lambda^i_{\mathbf{N}^\bullet_{V(f)}, (t,\mathbf{z})}(\mathbf{0})$ is defined for $1 \leq i \leq n-1$, i.e., we need to show that

$$\dim_{\mathbf{0}} \operatorname{supp} \phi_{z_{i-1}}[-1]\psi_{z_{i-2}}[-1] \cdots \psi_{z_1}[-1]\psi_t[-1]\mathbf{N}^\bullet_{V(f)} \leq 0.$$

From the short exact sequence of perverse sheaves

$$0 \to \mathbf{N}^\bullet_{V(f)} \to \mathbb{Z}^\bullet_{V(f)}[n] \to \pi_*\mathbb{Z}^\bullet_{\mathbb{D} \times \widetilde{V(f_0)}}[n] \to 0,$$

it follows that $\lambda^i_{\mathbf{N}^\bullet_{V(f)}, (t,\mathbf{z})}(\mathbf{0})$ is defined if $\lambda^i_{\mathbb{Z}^\bullet_{V(f)}[n], (t,\mathbf{z})}(\mathbf{0})$ is defined, by the triangle inequality for supports of perverse sheaves.

Since $(t, \mathbf{z})$ is an IPA-tuple for f at $\mathbf{0}$, Proposition 2.1.0.4 gives, for $1 \leq i \leq n-1$,

$$\dim_{\mathbf{0}} \operatorname{supp} \phi_{z_i}[-1]\mathbb{Z}^\bullet_{V(f,t,z_1,\cdots,z_{i-1})}[n-i] \leq 0.$$

Thus, away from $\mathbf{0}$, each of the comparison morphisms

$$\mathbb{Z}^\bullet_{V(f,t,z_1,\cdots,z_{i-1},z_i)}[n-i-1] \xrightarrow{\sim} \psi_{z_i}[-1]\mathbb{Z}^\bullet_{V(f,t,z_1,\cdots,z_{i-1})}[n-i]$$

is an isomorphism for $1 \leq i \leq n-1$. Consequently,

$$\dim_{\mathbf{0}} \operatorname{supp} \phi_{z_i}[-1]\mathbb{Z}^\bullet_{V(f,t,z_1,\cdots,z_{i-1})}[n-i] \leq 0$$

implies

$$\dim_{\mathbf{0}} \operatorname{supp} \phi_{z_{i-1}}[-1]\psi_{z_{i-2}}[-1]\cdots\psi_{z_1}[-1]\psi_t[-1]\mathbb{Z}^\bullet_{V(f)}[n] \leq 0,$$

and the claim follows.

□

Remark 2.3.0.14. In the wake of a recent paper [39] by David Massey, we can obtain a much simpler proof of the above result; one has the identification $\mathbf{N}^\bullet_{V(f)} \cong \ker\{\mathrm{id} - \widetilde{T}_f\}$ for hypersurfaces, where $\widetilde{T}_f$ is the Milnor monodromy automorphism on the vanishing cycles $\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]$. Consequently, $\mathbf{N}^\bullet_{V(f)}$ is a perverse subobject of the vanishing cycles $\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]$, and we obtain Theorem 2.3.0.13 by either the triangle inequality for microsupports, or the fact that characteristic polar multiplicities are additive on short exact sequences [45] from the fact that $\operatorname{supp}\mathbf{N}^\bullet_{V(f)} \subseteq \operatorname{supp}\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1] = \Sigma f$.

2.4 Milnor's Result and Beyond

We wish to express the Lê numbers of f_0 entirely in terms of data from the Lê numbers of f_{t_0} and the characteristic polar multiplicities of both $\mathbf{N}^\bullet_{V(f_0)}$ and $\mathbf{N}^\bullet_{V(f_{t_0})}$, for t_0 small and nonzero. The starting point is Proposition 2.1.0.10:

$$\begin{aligned}\lambda^0_{f_0,\mathbf{z}}(\mathbf{0}) &= \left(\Gamma^1_{f,t}\cdot V(t)\right)_{\mathbf{0}} + \lambda^1_{f,(t,\mathbf{z})}(\mathbf{0})\\ \lambda^i_{f_{t_0},\mathbf{z}}(\mathbf{0}) &= \lambda^{i+1}_{f,(t,\mathbf{z})}(\mathbf{0}),\end{aligned}$$

where $(t,\mathbf{z}) = (t, z_1, \cdots, z_n)$ is an IPA-tuple for f at $\mathbf{0}$. From Lemma 2.3.0.10, we have $\left(\Gamma^1_{f,t}\cdot V(t)\right)_{\mathbf{0}} = \lambda^0_{\mathbf{N}^\bullet_{V(f)},(t,\mathbf{z})}(\mathbf{0})$; we now have all our relevant data in terms of Lê numbers and characteristic polar multiplicities of $\mathbf{N}^\bullet_{V(f)}$. The goal is then to decompose this data into numerical invariants which refer only to the $t=0$ and $t\neq 0$ slices of $V(f)$.

So, in order to realize this goal, the next step is to decompose $\lambda^0_{\mathbf{N}^\bullet_{V(f)},(t,\mathbf{z})}(\mathbf{0})$ and $\lambda^i_{f,(t,\mathbf{z})}(\mathbf{0})$ for $i \geq 1$.

The 1-dimensional Lê number $\lambda^1_{f,(t,\mathbf{z})}(\mathbf{0})$ is the easiest; by the dynamic intersection property for proper intersections,

$$\begin{aligned}
\lambda^1_{f,(t,\mathbf{z})}(\mathbf{0}) &= \left(\Lambda^1_{f,(t,\mathbf{z})} \cdot V(t)\right)_{\mathbf{0}} \\
&= \sum_{p \in B_\epsilon \cap V(t-t_0)} \left(\Lambda^1_{f,(t,\mathbf{z})} \cdot V(t-t_0)\right)_p \\
&= \sum_{p \in B_\epsilon \cap V(t-t_0)} \lambda^0_{f_{t_0},\mathbf{z}}(p).
\end{aligned}$$

The approach for $\lambda^i_{f_0,\mathbf{z}}(\mathbf{0})$ for $i \geq 1$ is similar: we will use the fact that f is an IPA-deformation of f_0 to "move" around the origin in the $V(t)$ slice, and then use the dynamic intersection property.

Proposition 2.4.0.1. *If $(t,\mathbf{z}) = (t, z_1, \cdots, z_i)$ is an IPA-tuple for f at $\mathbf{0}$ for $i \geq 1$, the following equality of intersection numbers holds:*

$$\lambda^i_{f_0,\mathbf{z}}(\mathbf{0}) = \sum_{q \in B_\epsilon \cap V(t-t_0, z_1, z_2, \cdots, z_i)} \lambda^i_{f_{t_0},\mathbf{z}}(q)$$

where $0 < |t_0| \ll \epsilon \ll 1$

Proof. First, recall that $\lambda^i_{f_0,\mathbf{z}}(\mathbf{0}) = \left(\Lambda^i_{f_0,\mathbf{z}} \cdot V(z_1, \cdots, z_i)\right)_{\mathbf{0}}$, where $\Lambda^i_{f_0,\mathbf{z}}$ is the i-dimensional Lê cycle of f_0 with respect to $\mathbf{z}$ (see Appendix A, as well as [42]). For $i \geq 1$, we have

$$\Lambda^i_{f_0,\mathbf{z}} = \Lambda^{i+1}_{f,(t,\mathbf{z})} \cdot V(t),$$

so, by the dynamic intersection property,

$$\begin{aligned}
\lambda^i_{f_0,\mathbf{z}}(\mathbf{0}) &= \left(\Lambda^{i+1}_{f,(t,\mathbf{z})} \cdot V(t, z_1, \cdots, z_i)\right)_{\mathbf{0}} \\
&= \sum_{q \in B_\epsilon \cap V(t-t_0, z_1, z_2, \cdots, z_i)} \left(\Lambda^{i+1}_{f,(t,\mathbf{z})} \cdot V(t-t_0, z_1, z_2, \cdots, z_i)\right)_q \\
&= \sum_{q \in B_\epsilon \cap V(t-t_0, z_1, z_2, \cdots, z_i)} \left(\Lambda^i_{f_{t_0},\mathbf{z}} \cdot V(z_1, z_2, \cdots, z_i)\right)_q \\
&= \sum_{q \in B_\epsilon \cap V(t-t_0, z_1, z_2, \cdots, z_i)} \lambda^i_{f_{t_0},\mathbf{z}}(\mathbf{0}),
\end{aligned}$$

where the second equality follows from the equality of cycles $\Lambda^{i+1}_{f,\mathbf{z}} \cdot V(t-t_0) = \Lambda^i_{f_{t_0},\mathbf{z}}$. □

We can now state and prove our main result.

Theorem 2.4.0.2. *Suppose that* $\pi : (\mathbb{D} \times \widetilde{V(f_0)}, \{0\} \times S) \to (V(f), \mathbf{0})$ *is a one-parameter unfolding with an isolated instability of a parameterized hypersurface* $\operatorname{im} \pi_0 = V(f_0)$*. Suppose further that* $\mathbf{z} = (z_1, \cdots, z_n)$ *is chosen such that* $\mathbf{z}$ *is an IPA-tuple for* $f_0 = f_{|_{V(t)}}$ *at* $\mathbf{0}$*. Then, the following formulas hold for the Lê numbers of* f_0 *with respect to* $\mathbf{z}$ *at* $\mathbf{0}$*: for* $0 < |t_0| \ll \epsilon \ll 1$,

$$\lambda^0_{f_0,\mathbf{z}}(\mathbf{0}) = -\lambda^0_{\mathbf{N}^\bullet_{V(f_0)},\mathbf{z}}(\mathbf{0}) + \sum_{p \in B_\epsilon \cap V(t-t_0)} \left(\lambda^0_{f_{t_0},\mathbf{z}}(p) + \lambda^0_{\mathbf{N}^\bullet_{V(f_{t_0})},\mathbf{z}}(p) \right),$$

and, for $1 \leq i \leq n-2$,

$$\lambda^i_{f_0,\mathbf{z}}(\mathbf{0}) = \sum_{q \in B_\epsilon \cap V(t-t_0, z_1, z_2, \cdots, z_i)} \lambda^i_{f_{t_0},\mathbf{z}}(q).$$

In particular, the following relationship holds for $0 \leq i \leq n-2$*:*

$$\lambda^i_{f_0,\mathbf{z}}(\mathbf{0}) + \lambda^i_{\mathbf{N}^\bullet_{V(f_0)},\mathbf{z}}(\mathbf{0}) = \sum_{p \in B_\epsilon \cap V(t-t_0, z_1, z_2, \cdots, z_i)} \left(\lambda^i_{f_{t_0},\mathbf{z}}(p) + \lambda^i_{\mathbf{N}^\bullet_{V(f_{t_0})},\mathbf{z}}(p) \right)$$

Proof. By Proposition 2.1.0.10 and Proposition 2.4.0.1, it suffices to prove

$$\lambda^0_{\mathbf{N}^\bullet_{V(f)},(t,\mathbf{z})}(\mathbf{0}) = -\lambda^0_{\mathbf{N}^\bullet_{V(f_0)},\mathbf{z}}(\mathbf{0}) + \sum_{p \in B_\epsilon \cap V(t-t_0)} \lambda^0_{\mathbf{N}^\bullet_{V(f_{t_0})},\mathbf{z}}(p). \tag{2.4}$$

Since $(t, \mathbf{z})$ is an IPA-tuple for f at $\mathbf{0}$, Theorem 2.3.0.13 yields

$$\lambda^0_{\mathbf{N}^\bullet_{V(f_0)},\mathbf{z}}(\mathbf{0}) = \lambda^1_{\mathbf{N}^\bullet_{V(f)},(t,\mathbf{z})}(\mathbf{0}) - \lambda^0_{\mathbf{N}^\bullet_{V(f)},(t,\mathbf{z})}(\mathbf{0}),$$

where $\mathbf{N}^\bullet_{V(f_0)} \cong \mathbf{N}^\bullet_{V(f)_{|_{V(t)}}}[-1]$ (cf. Remark 2.2.0.4).

The main claim then follows by the dynamic intersection property for proper intersections applied to $\Lambda^1_{\mathbf{N}^\bullet_{V(f)},(t,\mathbf{z})}$ (see Remark 2.3.0.9):

$$\begin{aligned}
\lambda^1_{\mathbf{N}^\bullet_{V(f)},(t,\mathbf{z})}(\mathbf{0}) &= \left(\Lambda^1_{\mathbf{N}^\bullet_{V(f)},(t,\mathbf{z})} \cdot V(t) \right)_{\mathbf{0}} \\
&= \sum_{p \in B_\epsilon \cap V(t-t_0)} \left(\Lambda^1_{\mathbf{N}^\bullet_{V(f)},(t,\mathbf{z})} \cdot V(t-t_0) \right)_p \\
&= \sum_{p \in B_\epsilon \cap V(t-t_0)} \lambda^0_{\mathbf{N}^\bullet_{V(f_{t_0})},\mathbf{z}}(p),
\end{aligned}$$

for $0 < |t_0| \ll \epsilon \ll 1$.

Finally, we examine the relationship

$$\lambda^i_{f_0,\mathbf{z}}(\mathbf{0}) + \lambda^i_{\mathbf{N}^\bullet_{V(f_0)},\mathbf{z}}(\mathbf{0}) = \sum_{p \in B_\epsilon \cap V(t-t_0,z_1,z_2,\cdots,z_i)} \left(\lambda^i_{f_{t_0},\mathbf{z}}(p) + \lambda^i_{\mathbf{N}^\bullet_{V(f_{t_0})},\mathbf{z}}(p) \right).$$

For $i = 0$, this follows by a trivial rearrangement of the terms in our expression for $\lambda^0_{f_0,\mathbf{z}}(\mathbf{0})$. For $i \geq 1$, this is just Proposition 2.4.0.1 combined with Theorem 2.3.0.13 and the dynamic intersection property on $\lambda^i_{\mathbf{N}^\bullet_{V(f)},(t,\mathbf{z})}(\mathbf{0})$, as in Proposition 2.4.0.1 for $\lambda^i_{f,(t,\mathbf{z})}(\mathbf{0})$. □

Remark 2.4.0.3. The relationship

$$\lambda^i_{f_0,\mathbf{z}}(\mathbf{0}) + \lambda^i_{\mathbf{N}^\bullet_{V(f_0)},\mathbf{z}}(\mathbf{0}) = \sum_{p \in B_\epsilon \cap V(t-t_0,z_1,z_2,\cdots,z_i)} \left(\lambda^i_{f_{t_0},\mathbf{z}}(p) + \lambda^i_{\mathbf{N}^\bullet_{V(f_{t_0})},\mathbf{z}}(p) \right)$$

suggests a sort of "conserved quantity" between the sum of the Lê numbers of f_t and the characteristic polar multiplicities of $\mathbf{N}^\bullet_{V(f_t)}$ in one parameter deformations of parameterized hypersurfaces. It is a very interesting question to see how this relates to results in Chapter 3 and the isomorphism $\mathbf{N}^\bullet_{V(f)} \cong \ker\{\mathrm{id} - \widetilde{T}_f\}$ (see Section 1.2 and Section 3.4).

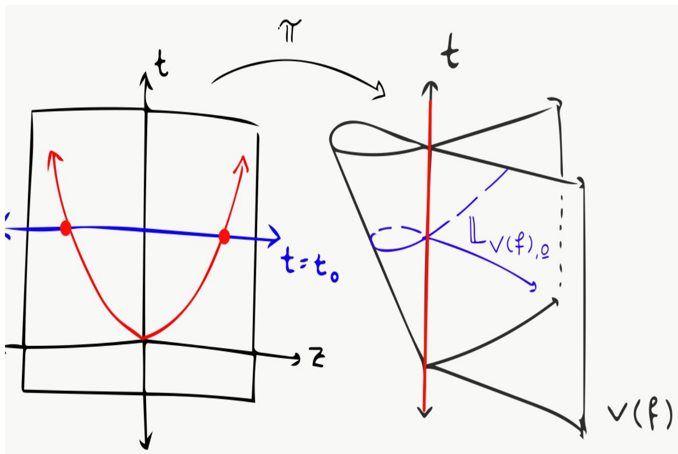
Example 2.4.0.4. We wish to examine Theorem 2.4.0.2 in the context of Milnor's double point formula, where $\pi : (\mathbb{D} \times \mathbb{C}, \{0\} \times S) \to (\mathbb{C}^3, \mathbf{0})$ parameterizes a deformation of the curve $V(f_0)$ into a curve $V(f_{t_0})$ with only double-point singularities. In this case, $\dim_{\mathbf{0}} \Sigma f_0 = 0$, so the only non-zero Lê number of f_0 is $\lambda^0_{f_0,z}(\mathbf{0})$, where z is any non-zero linear form on $\mathbb{C}^2$, and $\lambda^0_{f_0,z}(\mathbf{0}) = \mu_{\mathbf{0}}(f_0)$.

It is then an easy exercise to see that $\lambda^0_{\mathbf{N}^\bullet_{V(f_{t_0})},z}(p) = m(p) = |\pi^{-1}(p)| - 1$ for t_0 small (and possibly zero) and $p \in D$.

All together, this gives, by Theorem 2.4.0.2

$$\begin{aligned} \mu_{\mathbf{0}}(f_0) &= -(r-1) + \sum_{p \in B_\epsilon \cap V(t-t_0)} \left(\mu_p(f_{t_0}) + |\pi^{-1}(p)| - 1 \right) \\ &= 2\delta - r + 1, \end{aligned}$$

as there are δ double-points in the deformed curve $V(f_{t_0})$. We have thus recovered Milnor's original double-point formula for the Milnor number of a plane curve singularity. We picture this computation below.



Remark 2.4.0.5. In Lemma 2.2 of [56], David Mond also obtains the result that, for a stabilization of a plane curve singularity $V(f_0)$, one has

$$\mu_{\mathbf{0}}\left(t_{|_{V(f)}}\right) = \delta - r + 1,$$

where $\mu_{\mathbf{0}}\left(t_{|_{V(f)}}\right)$ is called the **image Milnor number** of the stabilization. It is an interesting question in general how one can relate the theory of map germs from $\mathbb{C}^n$ to $\mathbb{C}^{n+1}$ of finite $\mathcal{A}$-codimension (in Mather's nice dimensions ($n < 15$) and beyond) to our result Theorem 2.4.0.2.

Remark 2.4.0.6. Gaffney also generalizes the result $\mu_{\mathbf{0}}(t_{|_{V(f)}}) = \delta - r + 1$ in [13], although to the very different setting of maps $G : (\mathbb{C}^n, S) \to (\mathbb{C}^{2n}, \mathbf{0})$. In Theorem 3.2 and Corollary 3.3 of [13], this formula is derived in terms of the Segre number of dimension 0 of an ideal associated to the image multiple-point set D and the number of Whitney umbrellas of the composition of the map G with a generic projection to $\mathbb{C}^{2n-1}$.

In analogy to plane curve singularities deforming into node singularities, it is well-known (see, e.g., [57]), that for stabilizations of finitely determined maps $\pi_0 : (\mathbb{C}^2, S) \to (\mathbb{C}^3, \mathbf{0})$, the image surface $\operatorname{im}\pi_0 = V(f_0)$ splits into Cross Caps (i.e., Whitney Umbrellas), Triple Points, and A_1-singularities (i.e., nodes, which appear off the hypersurface on the relative polar curve). Unfortunately, detecting these invariants using characteristic polar multiplicities and Theorem 2.4.0.2 will have an unavoidable problem: we will also see points that belong to the **absolute polar curve** $\Gamma^1_{\mathbf{z}}(\Sigma f)$, which lie in the smooth part of Σf near $\mathbf{0}$, and are artifacts of our choice of linear forms $\mathbf{z}$ in calculating the characteristic polar multiplicities. For $\mathbf{z} = (t, z)$ a generic pair of linear forms on $\mathbb{C}^4$, the absolute polar curve of Σf at $\mathbf{0}$ is

$$\Gamma^1_{\mathbf{z}}(\Sigma f) = \overline{\Sigma\left((t,z)_{|\Sigma f}\right) - \Sigma(\Sigma f)}$$

(see [31], [70], but we instead index by dimension instead of codimension). Consequently, if $p \in \Gamma^1_{\mathbf{z}}(\Sigma f)\backslash\{\mathbf{0}\}$, we see that $\lambda^0_{\mathbf{N}^\bullet_{V(f_{t_0})},z}(p) \neq 0$ even if the stalk cohomology of $\mathbf{N}^\bullet_{V(f_{t_0})}$ is locally constant near p. We thus obtain the following result:

Theorem 2.4.0.7. *Suppose $\pi : (\mathbb{D}\times\mathbb{C}^2, \{\mathbf{0}\}\times S) \to (\mathbb{C}^4, \mathbf{0})$ is a one-parameter unfolding of a finitely-determined map germ $\pi_0 : (\mathbb{C}^2, S) \to (\mathbb{C}^3, \mathbf{0})$ parameterizing a surface $V(f_0) \subseteq \mathbb{C}^3$. Then,*

$$\lambda^0_{\mathbf{N}^\bullet_{V(f_0)},\mathbf{z}}(\mathbf{0}) = T + C - \delta + P$$

where T, C, δ, and P denote the number of triple points, cross caps, A_1-singularities appearing in a stable deformation of $V(f_0)$, respectively, and if $V(f) = \operatorname{im}\pi$, P denotes the number of intersection points of the absolute polar curve $\Gamma^1_{(t,z)}(\Sigma f)$ with a generic hyperplane $V(z)$ on $\mathbb{C}^4$ for which (t, z) is an IPA-tuple for f at $\mathbf{0}$.

Proof. This follows directly from Theorem 2.3.0.13, Remark 2.2.0.3, Lemma 2.3.0.10, and recalling that $\lambda^0_{\mathbf{N}^\bullet_{V(f_{t_0})},\mathbf{z}}(\mathbf{0}) = 1$ for both Whitney umbrellas and triple point singularities in $\mathbb{C}^3$ (see Example 2.3.0.12 and Example 2.3.0.8). $\square$

In fact, we can explicitly identify the Euler characteristic $\lambda^0_{\mathbf{N}^\bullet_{V(f_0)},\mathbf{z}}(\mathbf{0}) - \lambda^1_{\mathbf{N}^\bullet_{V(f_0)},\mathbf{z}}(\mathbf{0})$ using Theorem 2.4.0.7.

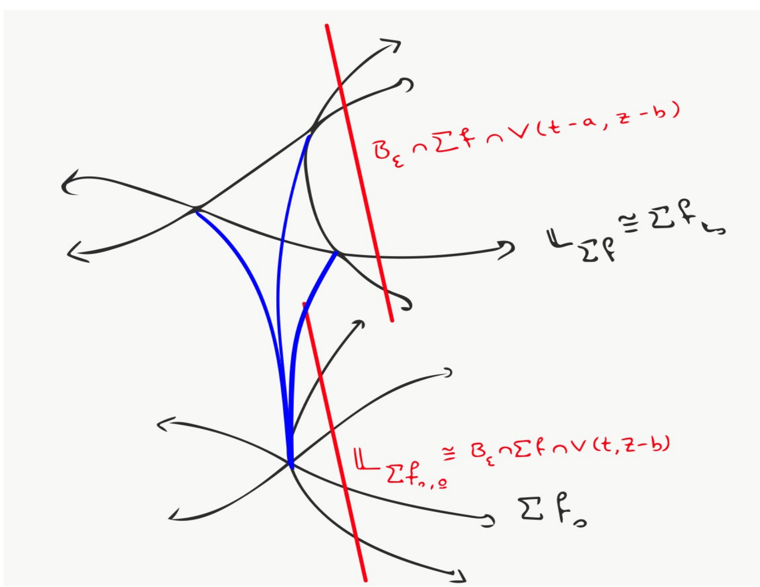
Corollary 2.4.0.8. *Let π_0, π, T, C, δ, and P be as in Theorem 2.4.0.7. Then, the following equalities hold:*

$$\lambda^0_{\mathbf{N}^\bullet_{V(f_0)},\mathbf{z}}(\mathbf{0}) - \lambda^1_{\mathbf{N}^\bullet_{V(f_0)},\mathbf{z}}(\mathbf{0}) = \chi(\mathbf{N}^\bullet_{V(f_0)})_\mathbf{0} = -|\pi_0^{-1}(\mathbf{0})| + 1 = C - T - \delta - \chi(F_{t_{|\Sigma f},\mathbf{0}}),$$

where $F_{t_{|\Sigma f},\mathbf{0}}$ denotes the complex link of Σf at $\mathbf{0}$.

Remark 2.4.0.9. Before we give the proof of Corollary 2.4.0.8 using derived category techniques, we will give a down-to-earth topological argument. The key idea in our proof is that one can compute the term P using constant $\mathbb{Z}$ coefficients instead of $\mathbf{N}^\bullet_{V(f)}$, since $\mathbf{N}^\bullet_{V(f)}$ generically has stalk cohomology $\mathbb{Z}$ along Σf for hypersurfaces $V(f)$ that are the image of finitely-determined map germs.

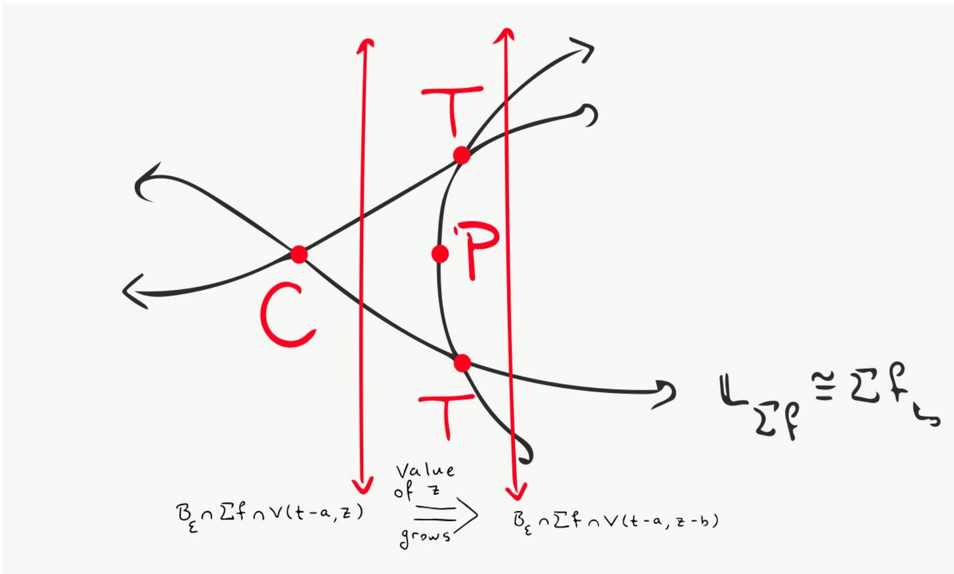
Proof. (topological argument) We compute the Euler characteristic of the pair $\chi(F_{t_{|\Sigma f},\mathbf{0}}, F_{z_{|\Sigma f_0},\mathbf{0}})$. This pair of subspaces makes sense, using the fact that f is an IPA-deformation of f_0, and the complex link $F_{z_{|\Sigma f_0},\mathbf{0}}$ of Σf_0 is a finite set of points, and their multiplicity is unchanged as one moves in the t direction away from the origin, pictured below:



Thus, we can identify $F_{z_{|\Sigma f_0},\mathbf{0}} = B_\epsilon \cap \Sigma f \cap V(t, z-b)$ with $B_\epsilon \cap \Sigma f \cap V(t-a, z-b)$ for $0 < |a| \ll |b| \ll \epsilon \ll 1$. Consequently, we can identify

$$\chi(F_{t_{|\Sigma f},\mathbf{0}}, F_{z_{|\Sigma f_0},\mathbf{0}}) = \chi(\phi_z[-1]\mathbb{Z}^\bullet_{F_{t_{|\Sigma f},\mathbf{0}}}[1])_\mathbf{0} = \sum_p \lambda^0_{\mathbb{Z}^\bullet_{\Sigma f_{t_0}}[1],z}(p).$$

As the value of z grows from 0 to b, we pick up cohomological contributions (in the form of a non-zero multiplicity $\lambda^0_{\mathbb{Z}^\bullet_{\Sigma f_{t_0}}[1],z}(p)$) as we pass through points of the curves of triple points, cross caps, and the absolute polar curve with respect to (t,z), pictured below:



At triple points, $\lambda^0_{\mathbb{Z}^\bullet_{\Sigma f_{t_0}}[1],z}(p) = 2$, and at cross caps $\lambda^0_{\mathbb{Z}^\bullet_{\Sigma f_{t_0}}[1],z}(p) = 0$. We count the contribution from the absolute polar curve as $P = \left(\Gamma^1_{(t,z)}(\Sigma f) \cdot V(z)\right)_{\mathbf{0}}$. Thus,

$$\begin{aligned} 2T + P = \chi(F_{t_{|\Sigma f},\mathbf{0}}, F_{z_{|\Sigma f_0},\mathbf{0}}) &= \chi(F_{t_{|\Sigma f},\mathbf{0}}) - \chi(F_{z_{|\Sigma f_0},\mathbf{0}}) \\ &= \chi(F_{t_{|\Sigma f},\mathbf{0}}) - \lambda^1_{\mathbf{N}^\bullet_{V(f_0)},z}(\mathbf{0}). \end{aligned}$$

Solving for P and plugging the resulting expression into Theorem 2.4.0.7 gives the result. $\square$

Proof. (perverse sheaves argument) We wish to better understand the contribution of the term P coming from the absolute polar curve of Σf appearing in Theorem 2.4.0.7. First, we note that these terms come from the 0-dimensional characteristic polar multiplicities $\lambda^0_{\mathbf{N}^\bullet_{V(f_{t_0})},\mathbf{z}}(p)$ in the expansion of $\lambda^1_{\mathbf{N}^\bullet_{V(f)},(t,z)}(\mathbf{0})$, where p is a smooth point of Σf in the $V(t-t_0)$ slice. Since the transverse singularity type of the image of a finitely-determined map is always that of a Morse function, the stalk cohomology of $\mathbf{N}^\bullet_{V(f)}$ is $\mathbb{Z}$ at all smooth points of Σf. Consequently, we can calculate P using the constant sheaf $\mathbb{Z}^\bullet_{\Sigma f}[2]$ in place of $\mathbf{N}^\bullet_{V(f)}$.

However, $\mathbb{Z}^\bullet_{\Sigma f}[2]$ is not necessarily a perverse sheaf. To deal with this, note that, for all $t_0 \neq 0$, the restriction $\left(\mathbb{Z}^\bullet_{\Sigma f}[2]\right)_{|_{V(t-t_0)}} \cong \mathbb{Z}^\bullet_{\Sigma f_{t_0}}[1]$ is a perverse sheaf (the shifted constant sheaf on a curve is always perverse), and therefore $\psi_t[-1]\mathbb{Z}^\bullet_{\Sigma f}[2]$ is perverse.

We then examine Euler characteristics at the origin of the distinguished triangle

$$\left(\psi_t[-1]\mathbb{Z}^\bullet_{\Sigma f}[2]\right)_{|_{V(z)}}[-1] \to \psi_z[-1]\psi_t[-1]\mathbb{Z}^\bullet_{\Sigma f}[2] \to \phi_z[-1]\psi_t[-1]\mathbb{Z}^\bullet_{\Sigma f}[2] \xrightarrow{+1}, \tag{2.5}$$

where $\psi_z[-1]\psi_t[-1]\mathbb{Z}^\bullet_{\Sigma f}[2]$ and $\phi_z[-1]\psi_t[-1]\mathbb{Z}^\bullet_{\Sigma f}[2]$ are perverse sheaves for which $\mathbf{0}$ is an isolated point in their support. By Definition 2.3.0.1,

$$\chi(\phi_z[-1]\psi_t[-1]\mathbb{Z}^\bullet_{\Sigma f}[2])_\mathbf{0} = \operatorname{rank} H^0(\phi_z[-1]\psi_t[-1]\mathbb{Z}^\bullet_{\Sigma f}[2])_\mathbf{0} = \lambda^1_{\mathbb{Z}^\bullet_{\Sigma f}[2],(t,z)}(\mathbf{0}).$$

To calculate $\chi(\psi_z[-1]\psi_t[-1]\mathbb{Z}^\bullet_{\Sigma f}[2])_\mathbf{0}$, note that $\dim_\mathbf{0} \operatorname{supp} \phi_t[-1]\mathbb{Z}^\bullet_{\Sigma f}[2] \leq 0$ (since f is an IPA-deformation of $f_{|_{V(t)}}$ at $\mathbf{0}$) implies $\psi_z[-1]\phi_t[-1]\mathbb{Z}^\bullet_{\Sigma f}[2] = 0$, and so

$$\psi_z[-1]\mathbb{Z}^\bullet_{\Sigma f_0}[1] \xrightarrow{\sim} \psi_z[-1]\psi_t[-1]\mathbb{Z}^\bullet_{\Sigma f}[2].$$

Thus, $\chi(\psi_z[-1]\psi_t[-1]\mathbb{Z}^\bullet_{\Sigma f}[2])_\mathbf{0} = \chi(\psi_z[-1]\mathbb{Z}^\bullet_{\Sigma f_0}[1])_\mathbf{0} = \lambda^1_{\mathbb{Z}^\bullet_{\Sigma f_0}[1],z}(\mathbf{0})$. It is easy to see that $\lambda^1_{\mathbb{Z}^\bullet_{\Sigma f_0}[1],z}(\mathbf{0}) = \lambda^1_{\mathbf{N}^\bullet_{V(f_0)},z}(\mathbf{0})$, since the transverse singularity type of Σf_0 is that of a Morse function.

Finally, we see that $\chi(\left(\psi_t[-1]\mathbb{Z}^\bullet_{\Sigma f}[2]\right)_{|_{V(z)}}[-1])_\mathbf{0} = \chi(F_{t_{|_{\Sigma f}},\mathbf{0}})$, and we obtain the following formula from taking the Euler characteristic of (2.5):

$$\chi(F_{t_{|_{\Sigma f}},\mathbf{0}}) - \lambda^1_{\mathbf{N}^\bullet_{V(f_0)},z}(\mathbf{0}) + \lambda^1_{\mathbb{Z}^\bullet_{\Sigma f}[2],(t,z)}(\mathbf{0}) = 0. \tag{2.6}$$

Using the dynamic intersection property,

$$\lambda^1_{\mathbb{Z}^\bullet_{\Sigma f}[2],(t,z)}(\mathbf{0}) = \sum_{p \in B_\epsilon \cap V(t-t_0)} \lambda^0_{\mathbb{Z}^\bullet_{\Sigma f_{t_0}}[1],z}(p) = 2T + P,$$

since $\lambda^0_{\mathbb{Z}^\bullet_{\Sigma f_{t_0}}[1],z}(p) = 2$ when p is a triple point singularity, and $\lambda^0_{\mathbb{Z}^\bullet_{\Sigma f_{t_0}}[1],z}(p) = 0$ when p is a cross-cap singularity. The remaining terms, as in Theorem 2.4.0.7, come from the absolute polar curve of Σf with respect to $V(z)$. Consequently, we can solve for P using (2.6)

$$P = \lambda^1_{\mathbf{N}^\bullet_{V(f_0)},z}(\mathbf{0}) - \chi(F_{t_{|_{\Sigma f}},\mathbf{0}}) - 2T.$$

Plugging this expression for P into Theorem 2.4.0.7 tells us

$$\begin{aligned}\lambda^0_{\mathbf{N}^\bullet_{V(f_0)},z}(\mathbf{0}) &= T + C - \delta + P \\ &= T + C - \delta + \lambda^1_{\mathbf{N}^\bullet_{V(f_0)},z}(\mathbf{0}) - \chi(F_{t_{|\Sigma f},\mathbf{0}}) - 2T\end{aligned}$$

and so

$$\chi(\mathbf{N}^\bullet_{V(f_0)})_{\mathbf{0}} = \lambda^0_{\mathbf{N}^\bullet_{V(f_0)},z}(\mathbf{0}) - \lambda^1_{\mathbf{N}^\bullet_{V(f_0)},z}(\mathbf{0}) = C - T - \delta - \chi(F_{t_{|\Sigma f},\mathbf{0}}).$$

Finally, the Corollary follows from the fact that $\mathbf{N}^\bullet_{V(f_0)}$ has stalk cohomology concentrated in degree -1 (by Theorem 1.1.1.4 and Remark 2.3.0.7) □

Remark 2.4.0.10. If $V(f_0)$ is itself a $\mathbb{Q}$-homology manifold, then $\mathbf{N}^\bullet_{V(f_0)} = 0$. In this case, Theorem 2.4.0.7 tells us that, in a stabilization $V(f)$ of $V(f_0)$, we have

$$\chi(F_{t_{|\Sigma f},\mathbf{0}}) = C - T - \delta.$$

Remark 2.4.0.11. In the case of finitely-determined maps $F : (\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}^3, \mathbf{0})$ of the form $F(t, z^2, F_3(t, z))$, $\operatorname{im} F = V(f)$ defines a surface whose singular locus Σf is an isolated complete intersection singularity (ICIS) by results of Mond and Pellikaan (e.g., Prop. 2.2.4 of [58]). In this case, the results of [12] apply, and we can recover Gaffney's formula (Proposition 2.4) for the 0-dimensional Lê number of f at $\mathbf{0}$

$$\lambda^0_{f,\mathbf{z}}(\mathbf{0}) = \delta + 2C + e(JM(\Sigma f)).$$

where δ (resp., C) is the number of A_1-singularities (resp., cross caps) appearing in a stabilization of F, and $e(JM(\Sigma f))$ is the Buchbaum-Rim multiplicity of the Jacobian Module of $\Sigma f = D$.

It is a very interesting question to see what formulas might arise from Theorem 2.4.0.2 when one works outside of Mather's nice dimensions; for $n \geq 15$, one can no longer approximate a finitely determined map with stable maps, but the relationship in Theorem 2.4.0.2 still holds.

Chapter 3

Some Hodge Theoretic Aspects of Parameterized Spaces

This chapter arose from an innocuous question posed to us by a referee of [21]: **when is $\mathbf{N}_X^\bullet$ a semi-simple perverse sheaf, so that $\mathbb{Q}_X^\bullet[n]$ is an extension of semi-simple perverse sheaves?**

We were able to provide a somewhat "unsatisfying" partial answer (below) to this question in Remark 2.5 of [21] in response to this referee; later, we provided a complete answer that eventually became [22]. It is this complete answer, and the details involved, that we will present in this chapter.

When X is a parameterized space, both $\mathbf{I}_X^\bullet$ and $\mathbf{N}_X^\bullet$ are, in fact, just sheaves (up to a shift); moreover, the short exact sequence of perverse sheaves

$$0 \to \mathbf{N}_X^\bullet \to \mathbb{Q}_X^\bullet[n] \to \mathbf{I}_X^\bullet \to 0$$

can be rewritten as a short exact sequence of (constructible) sheaves

$$0 \to \mathbb{Q}_X \to H^{-n}(\mathbf{I}_X^\bullet) \to H^{-n+1}(\mathbf{N}_X^\bullet) \to 0.$$

One can then find a Whitney stratification $\mathfrak{S}$ of X for which the sets X_k (see Formula 1.6) are locally finite unions of strata for all k. Then, for each stratum $S \subset X_k$, the monodromy of the local system $\mathbf{N}^\bullet_{X|_S}$ is determined by the monodromy of the set $\pi^{-1}(p)$ for $p \in S$; since this is a finite set with k elements, it follows immediately that $\mathbf{N}^\bullet_{X|_S}$ is semi-simple **as a local system on** S (since the monodromy action is semi-simple).

Since $\mathbf{N}^\bullet_{X|_S}$ is semi-simple as a local system for any stratum $S \subset X_k$, is $\mathbf{N}^\bullet_X$ semi-simple as a perverse sheaf? If one has a Whitney stratification of X for which the sets X_k are finite unions of strata, and for which the subset $D = \operatorname{supp} \mathbf{N}^\bullet_X$ is a union of *closed strata*, then the above argument demonstrates (together with Section 2.4 of [67]) that $\mathbf{N}^\bullet_X$ is semi-simple as a perverse sheaf. In general, however, this fails to be the case (see Subsection 1.1.2).

More generally, when $\mathbb{Q}^\bullet_X[n]$ is a perverse sheaf, one may use the general machinery of mixed Hodge modules developed by M. Saito (see [66], page 325, formula (4.5.9)) to obtain an isomorphism of perverse sheaves

$$\mathrm{Gr}^W_n \mathbb{Q}^\bullet_X[n] \xrightarrow{\sim} \mathbf{I}^\bullet_X. \tag{3.1}$$

underlying the corresponding isomorphism of mixed Hodge modules. Since $\dim_{\mathbf{0}} X = n$, the weight filtration on $\mathbb{Q}^\bullet_X[n]$ terminates after degree n, so that $W_n\mathbb{Q}^\bullet_X[n] \cong \mathbb{Q}^\bullet_X[n]$. Consequently, the above isomorphism yields a short exact sequence

$$0 \to W_{n-1}\mathbb{Q}^\bullet_X[n] \to \mathbb{Q}^\bullet_X[n] \to \mathbf{I}^\bullet_X \to 0$$

of perverse sheaves on X, implying $\mathbf{N}^\bullet_X \cong W_{n-1}\mathbb{Q}^\bullet_X[n]$. From this identification, it follows that $\mathbf{N}^\bullet_X$ is semi-simple as a perverse sheaf provided that the induced weight filtration $W_i\mathbb{Q}^\bullet_X[n]$ of $W_{n-1}\mathbb{Q}^\bullet_X[n] \cong \mathbf{N}^\bullet_X$ for $i < n$ is concentrated in one degree $k < n$, i.e., $W_i\mathbb{Q}^\bullet_X[n] = 0$ for $i < k$ and $W_i\mathbb{Q}^\bullet_X[n] \cong W_k\mathbb{Q}^\bullet_X[n]$ for $k < i < n$. Then, $\mathbf{N}^\bullet_X \cong \mathrm{Gr}^W_k \mathbb{Q}^\bullet_X[n]$ underlies a pure polarizable Hodge module, which is therefore by construction a semi-simple perverse sheaf. Identifying when this happens is quite involved and will be the focus of this chapter, with a complete answer proved in Theorem 3.2.0.6 and Corollary 3.2.0.7.

3.1 Introduction

Let $\mathcal{U}$ be an open neighborhood of the origin in $\mathbb{C}^N$, and let $X \subseteq \mathcal{U}$ be a purely n dimensional reduced complex analytic space on which $\mathbb{Q}^\bullet_X[n]$ is perverse.

By shrinking $\mathcal{U}$ if necessary, the perverse sheaf $\mathbb{Q}^\bullet_X[n]$ underlies a graded-polarizable mixed Hodge module (Prop 2.19, Prop 2.20, [66]) of weight $\leq n$. Moreover, by Saito's theory of (graded polarizable) mixed Hodge modules in the local complex analytic context, the perverse cohomology objects of the usual sheaf functors naturally lift to cohomology functors in the context of (graded polarizable) mixed Hodge modules (but not on their derived category level as in the algebraic context as in Section 4 of [66]). As above in Formula 3.1, the quotient morphism $\mathbb{Q}^\bullet_X[n] \to \mathbf{I}^\bullet_X$ induces an isomorphism

$$\operatorname{Gr}^W_n \mathbb{Q}^\bullet_X[n] \cong \mathbf{I}^\bullet_X;$$

consequently, the fundamental short exact sequence (1.1) identifies the comparison complex $\mathbf{N}^\bullet_X$ with $W_{n-1}\mathbb{Q}^\bullet_X[n]$. In this chapter, we explicitly identify the graded piece $\operatorname{Gr}^W_{n-1} \mathbf{N}^\bullet_X = \operatorname{Gr}^W_{n-1} \mathbb{Q}^\bullet_X[n]$ in the case where X is a parameterized space, and give concrete computations of $W_0\mathbb{Q}^\bullet_X[n]$ in the case where $X = V(f)$ is a parameterized surface in $\mathbb{C}^3$.

Let ΣX denote the singular locus of X, and let $i : \Sigma X \hookrightarrow X$. We can then find a smooth, Zariski open dense subset $\mathcal{W} \subseteq \Sigma X$ over which the normalization map restricts to a covering projection $\hat{\pi} : \pi^{-1}(\mathcal{W}) \to \mathcal{W} \subseteq \Sigma X$ (see Section 6.2, [16]). Let $l : \mathcal{W} \hookrightarrow \Sigma X$ and $m : \Sigma X \backslash \mathcal{W} \hookrightarrow \Sigma X$ denote the respective open and closed inclusion maps. Let $\hat{m} := i \circ m$, $\hat{l} := i \circ l$. Note that $\dim_{\mathbf{0}} \Sigma X \backslash \mathcal{W} \leq n-2$, as it is the complement of a Zariski open set (we will need this later in Proposition 3.2.0.2).

Example 3.1.0.1. Consider the Whitney umbrella $V(f) \subseteq \mathbb{C}^3$ with $f(x,y,t) = y^2 - x^3 - tx^2$. Then, the normalization of $V(f)$ is smooth, and given by the map $\pi(t,u) = (u^2 - t, u(u^2 - t), t)$.

The critical locus of f is $\Sigma f = V(x, y)$, and it is easy to see that over $\Sigma f \backslash \{\mathbf{0}\}$, π is a 2-to-1 covering map; thus, we set $\mathcal{W} = \Sigma f \backslash \{\mathbf{0}\}$.

Example 3.1.0.2. Suppose $V(f) \subseteq \mathbb{C}^3$ is a parameterized surface. Then, it is easy to see that $\mathcal{W} = \Sigma f \backslash \{\mathbf{0}\}$; this follows from the fact that $\mathbf{I}^\bullet_{V(f)_{|\Sigma f}}$ is is constructible with respect to the Whitney stratification $\{\Sigma f \backslash \{\mathbf{0}\}, \{\mathbf{0}\}\}$ of Σf, along with the description of the stalk cohomology of $\mathbf{I}^\bullet_{V(f)}$ given by the isomorphism $\mathbf{I}^\bullet_{V(f)} \cong \pi_* \mathbb{Q}^\bullet_{\widetilde{V(f)}}[2]$, where $\pi : \widetilde{V(f)} \to V(f)$ is the normalization of $V(f)$.

We will examine this setting in more detail in Section 3.3

Our main result of this chapter is the following.

Theorem 3.1.0.3 (Theorem 3.2.0.6)**.** *Suppose X is a parameterized space. Then, there is an isomorphism $\mathrm{Gr}^W_{n-1} i^* \mathbf{N}^\bullet_X \cong \mathbf{I}^\bullet_{\Sigma X}(\hat{l}^* \mathbf{N}^\bullet_X)$, so that the short exact sequence of perverse sheaves on X*

$$0 \to m_* {}^p H^0(m^! i^* \mathbf{N}^\bullet_X) \to i^* \mathbf{N}^\bullet_X \to \mathbf{I}^\bullet_{\Sigma X}(\hat{l}^* \mathbf{N}^\bullet_X) \to 0$$

identifies $W_{n-2} i^* \mathbf{N}^\bullet_X \cong m_* {}^p H^0(m^! i^* \mathbf{N}^\bullet_X)$. Here, $\mathbf{I}^\bullet_{\Sigma X}(\hat{l}^* \mathbf{N}^\bullet_X)$ denotes the intermediate extension of the perverse sheaf $\hat{l}^* \mathbf{N}^\bullet_X$ to all of ΣX, and ${}^p H^0(-)$ denotes the 0-th perverse cohomology functor.

Since the map $i : \Sigma X \hookrightarrow X$ is a closed inclusion, it preserves weights. Moreover, the support of $\mathbf{N}^\bullet_X$ is contained in the singular locus ΣX, and so $i_* i^* \mathbf{N}^\bullet_X \cong \mathbf{N}^\bullet_X$. Consequently, we have the following.

Corollary 3.1.0.4 (Corollary 3.2.0.7, Theorem 3.2.0.8)**.** *Suppose X is a parameterized space. Then, there are isomorphisms*

$$\mathrm{Gr}^W_{n-1} \mathbb{Q}^\bullet_X[n] \cong \mathrm{Gr}^W_{n-1} i^* \mathbf{N}^\bullet_X \cong i_* \mathbf{I}^\bullet_{\Sigma X}(l^* \mathbf{N}^\bullet_X),$$

and

$$W_{n-2} \mathbb{Q}^\bullet_X[n] \cong W_{n-2} i^* \mathbf{N}^\bullet_X \cong \hat{m}_* {}^p H^0(m^! i^* \mathbf{N}^\bullet_X) \cong m'_* \ker\{\phi_g[-1] i^* \mathbf{N}^\bullet_X \xrightarrow{\mathrm{var}} \psi_g[-1] i^* \mathbf{N}^\bullet_X\},$$

where g is any complex analytic function on ΣX such that $V(g)$ contains $\Sigma X \backslash \mathcal{W}$, but does not contain any irreducible component of ΣX, and $m' : V(g) \hookrightarrow \Sigma X$ is the closed inclusion.

In the case where $X = V(f)$ is a parameterized surface in $\mathbb{C}^3$, we explicitly compute $W_0\mathbb{Q}^\bullet_{V(f)}[2]$; the vanishing of this perverse sheaf places strong constraints on the topology of the singular set Σf of $V(f)$; we will see this later in Theorem 3.3.0.1.

3.2 General Case for Parameterized Spaces

In this section, we first prove a general result, Lemma 3.2.0.1, about perverse sheaves that will allow us to construct the short exact sequence mentioned in Theorem 3.2.0.6, and that $\mathbf{N}^\bullet_X$ satisfies the hypotheses of this lemma provided that X is parameterized. Then, we examine the weight filtration on $\mathbf{I}^\bullet_{\Sigma X}(\hat{l}^*\mathbf{N}^\bullet_X)$ and show that it underlies a polarizable Hodge module of weight $n-1$ in Proposition 3.2.0.5. With all this, we can state and prove Theorem 3.2.0.6 and Corollary 3.2.0.7.

Recall the category of perverse sheaves $\mathrm{Perv}(X)$ is the Abelian subcategory of the bounded derived category of sheaves of $\mathbb{Q}$-vector spaces with $\mathbb{C}$-constructible cohomology $D^b_c(X)$ given by the heart of the *perverse t-structure*, $\mathrm{Perv}(X) = {}^pD^{\leq 0}(X) \cap {}^pD^{\geq 0}(X)$. Here,

- $\mathbf{P}^\bullet \in {}^pD^{\leq 0}(X)$ if $\mathbf{P}^\bullet$ satisfies the *support condition*: for all $k \in \mathbb{Z}$,

$$\dim_{\mathbb{C}} \operatorname{supp} H^k(\mathbf{P}^\bullet) \leq -k.$$

- $\mathbf{P}^\bullet \in {}^pD^{\geq 0}(X)$ if $\mathcal{D}\mathbf{P}^\bullet$ satisfies the support condition, where $\mathcal{D}$ denotes the Verdier duality functor. This is known as the *cosupport condition.*

.

The following lemma is due to the author and David Massey.

Lemma 3.2.0.1. *Suppose X is a complex analytic space, $\mathbf{P}^\bullet$ a perverse sheaf on X, $l : \mathcal{W} \hookrightarrow X$ a Zariski open subset and $m : Z = X \backslash \mathcal{W} \hookrightarrow X$ its closed analytic complement. Then, if $m^*[-1]\mathbf{P}^\bullet \in {}^pD^{\leq 0}(Z)$, there is a short exact sequence*

$$0 \to m_*{}^pH^0(m^!\mathbf{P}^\bullet) \to \mathbf{P}^\bullet \to \mathbf{I}^\bullet_X(l^*\mathbf{P}^\bullet) \to 0$$

of perverse sheaves on X, where $\mathbf{I}^\bullet_X(l^*\mathbf{P}^\bullet) := \operatorname{im}{}^pH^0(l_!l^*\mathbf{P}^\bullet \to l_*l^*\mathbf{P}^\bullet)$ denotes the intermediate extension of $l^*\mathbf{P}^\bullet$ to all of X.

Proof. The natural morphism ${}^pH^0(l_!l^*\mathbf{P}^\bullet) \to {}^pH^0(l_*l^*\mathbf{P}^\bullet)$ factors as

$${}^pH^0(l_!l^*\mathbf{P}^\bullet) \xrightarrow{\alpha} \mathbf{P}^\bullet \xrightarrow{\beta} {}^pH^0(l_*l^*\mathbf{P}^\bullet).$$

From the other natural distinguished triangle associated to the pair of subsets $\mathcal{W}$ and Z,

$$l_!l^*\mathbf{P}^\bullet \to \mathbf{P}^\bullet \to m_*m^*\mathbf{P}^\bullet \xrightarrow{+1},$$

we see that surjectivity of α follows from the vanishing of

$${}^pH^0(m_*m^*\mathbf{P}^\bullet) \cong m_*{}^pH^0(m^*\mathbf{P}^\bullet).$$

By assumption, $m^*[-1]\mathbf{P}^\bullet \in {}^pD^{\leq 0}(Z)$, so that ${}^pH^k(m^*[-1]\mathbf{P}^\bullet) = 0$ for all $k > 0$. Thus,

$${}^pH^0(m^*\mathbf{P}^\bullet) \cong {}^pH^1(m^*[-1]\mathbf{P}^\bullet) = 0;$$

hence, α is surjective, and we have $\operatorname{im}\beta = \operatorname{im}(\beta \circ \alpha) \cong \mathbf{I}^\bullet_X(l^*\mathbf{P}^\bullet)$. We then obtain the isomorphism $\mathbf{I}^\bullet_X(l^*\mathbf{P}^\bullet) \cong \operatorname{im}\{\mathbf{P}^\bullet \to {}^pH^0(l_*l^*\mathbf{P}^\bullet)\}$.

Finally, the result follows from the long exact sequence in perverse cohomology associated to the distinguished triangle

$$m_*m^!\mathbf{P}^\bullet \to \mathbf{P}^\bullet \to l_*l^*\mathbf{P}^\bullet \xrightarrow{+1},$$

after noting that $m_*m^!\mathbf{P}^\bullet \in {}^pD^{\geq 0}(X)$ and $l_*l^*\mathbf{P}^\bullet \in {}^pD^{\geq 0}(X)$, (see, e.g., Proposition 10.3.3 of [25], or Theorem 5.2.4 of [6]). $\square$

From the introduction, let ΣX denote the singular locus of X, and let $i : \Sigma X \hookrightarrow X$. We can then find a smooth, dense Zariski open dense subset $\mathcal{W} \subseteq \Sigma X$ over which the normalization map restricts to a covering projection $\hat{\pi} : \pi^{-1}(\mathcal{W}) \to \mathcal{W} \subseteq \Sigma X$ (see Section 6.2, [16]). Let $l : \mathcal{W} \hookrightarrow \Sigma X$ and $m : \Sigma X \backslash \mathcal{W} \hookrightarrow \Sigma X$ denote the respective open and closed inclusion maps. Let $\hat{m} := i \circ m$, $\hat{l} := i \circ l$. Note that $\dim_{\mathbf{0}} \Sigma X \backslash \mathcal{W} \leq n-2$, as it is the complement of a Zariski open set.

Proposition 3.2.0.2. *If X is a parameterized space, then $\hat{m}^*[-1]\mathbf{N}^\bullet_X \in {}^pD^{\leq 0}(\Sigma X \backslash \mathcal{W})$.*

Proof. We wish to show that for all $k \in \mathbb{Z}$,

$$\dim_{\mathbb{C}} \operatorname{supp} H^k(\hat{m}^*[-1]\mathbf{N}^\bullet_X) \leq -k.$$

However, since X is parameterized, $\operatorname{supp} H^k(\hat{m}^*[-1]\mathbf{N}^\bullet_X)$ is non-empty only for $k-1 = -n+1$ by Theorem 1.1.1.4, i.e., when $k = -n+2$. In this degree, the support is equal to $\Sigma X \backslash \mathcal{W}$. Since this set is the complement of a Zariski open dense subset of ΣX,

$$\dim_{\mathbb{C}} \operatorname{supp} H^{-n+2}(\hat{m}^*\mathbf{N}^\bullet_X) \leq n-2,$$

as desired. $\square$

Remark 3.2.0.3. For surfaces $X = V(f)$ with curve singularities, $\hat{m}^*[-1]\mathbf{N}^\bullet_{V(f)} \in {}^pD^{\leq 0}(\Sigma f \backslash \mathcal{W})$ if and only if $V(f)$ is a parameterized space. To see this, first note by Example 3.1.0.2 that $\hat{m}$ is the inclusion of a point, and thus ${}^pD^{\leq 0}(\Sigma f \backslash \mathcal{W}) = D^{\leq 0}(\Sigma f \backslash \mathcal{W})$, i.e., the perverse t-structure on a point is the standard t-structure. Hence, $\hat{m}^*[-1]\mathbf{N}^\bullet_{V(f)} \in {}^pD^{\leq 0}(\Sigma f \backslash \mathcal{W})$ if and only if $H^k(\mathbf{N}^\bullet_{V(f)})_{\mathbf{0}} = 0$ for $k > -1$. For surfaces with curve singularities, this implies that the stalk cohomology of $\mathbf{N}^\bullet_{V(f)}$ at the origin is concentrated in degree -1. By Theorem 1.1.1.4, this is equivalent to $V(f)$ being a parameterized space.

In general, $\hat{m}^*[-1]\mathbf{N}^\bullet_X \in {}^pD^{\leq 0}(\Sigma X \backslash \mathcal{W})$ places strict constraints on the possible cohomology groups of the *real link* of X, denoted $K_{X,p}$, at different points $p \in \Sigma X$, i.e., the intersection of X with a sphere of sufficiently small radius at p.

Remark 3.2.0.4. Generically along an irreducible component C of ΣX, $\mathbf{N}^\bullet_X$ is isomorphic to a local system $\hat{l}^*(\mathbf{N}^\bullet_{X|_C})$ in degree $-n+1$, and in that degree, we have

$$H^{-n+1}(\mathbf{N}^\bullet_X)_p \cong \widetilde{IH}^0(B_\epsilon(p) \cap X; \mathbb{Q})$$

where $\widetilde{IH}^k$ denotes reduced intersection cohomology (with topological indexing, as in [17]). This description follows immediately from the fundamental short exact sequence (1.1). Since $\mathbf{I}^\bullet_X \cong \pi_* \mathbb{Q}^\bullet_{\widetilde{X}}[n]$, this reduced intersection cohomology is actually just

$$\widetilde{IH}^0(B_\epsilon(p) \cap X; \mathbb{Q}) \cong \widetilde{H}^0(K_{\widetilde{X}, \pi^{-1}(p)}; \mathbb{Q}),$$

where

$$K_{\widetilde{X}, \pi^{-1}(p)} = \bigcup_{q \in \pi^{-1}(p)} K_{\widetilde{X}, q}.$$

Since $\widetilde{X}$ is normal (and thus locally irreducible) it is clear that one has $H^0(K_{\widetilde{X},q}; \mathbb{Q}) \cong \mathbb{Q}$ for all $q \in \widetilde{X}$. After noting that $H^{-n}(\mathbf{I}^\bullet_X)_p = IH^0(K_{X,p})$ (that is, intersection cohomology of $K_{X,p}$ with topological indexing), $H^{-n}(\mathbf{I}^\bullet_X)_p$ has a pure Hodge structure of weight 0 (see, e.g., A. Durfee and M. Saito [7]).

Proposition 3.2.0.5. *Let C be an irreducible component of ΣX at $\mathbf{0}$. Then, $\hat{l}^*(\mathbf{N}^\bullet_{X|_C})$ underlies a polarizable variation of Hodge structure of weight* 0*.*

Consequently, $\mathbf{I}^\bullet_{\Sigma X}(\hat{l}^*\mathbf{N}^\bullet_X)$ underlies a polarizable Hodge module of weight $n-1$ on ΣX.

Proof. Since $\hat{l}^*\mathbf{N}^\bullet_X$ underlies a mixed Hodge module whose underlying perverse sheaf is a local system (up to a shift) on the complex manifold $\mathcal{U}$, this local system underlies an admissable graded polarizable variation of mixed Hodge structures on $\mathcal{U}$ by Theorem 3.27 of [66].

To show that this mixed Hodge structure is pure of weight zero, we can check on stalks at points $p \in \mathcal{U}$. Let $i_p : \{p\} \hookrightarrow \mathcal{U}$; then, the stalk cohomology $H^k(-)_p$ agrees with perverse cohomology ${}^pH^k(i_p^*)$. So, applying $H^k(i_p^*)$ on the level of mixed Hodge modules to the fundamental short exact sequence (1.1), we get (by Proposition 2.19, Proposition 2.20, and Theorem 3.9 of [66]) a short exact sequence in the category of graded polarizable mixed Hodge structures, whose underlying sequence of vector spaces is

$$0 \to \mathbb{Q}_{\{p\}} \to H^{-n}(\mathbf{I}^\bullet_X)_p \to H^{-n+1}(\mathbf{N}^\bullet_X)_p \to 0. \tag{3.2}$$

However, $\pi : \widetilde{X} \to X$ is a finite map, and therefore exact for the perverse t-structure (and mixed Hodge modules), with

$$H^{-n}(\mathbf{I}^\bullet_X)_p \cong H^{-n}(\pi_* \mathbb{Q}^\bullet_{\widetilde{X}}[n])_p \cong \bigoplus_{y \in \pi^{-1}(p)} \mathbb{Q}_{\{y\}}.$$

Since this stalk is pure of weight zero, the surjection in (3.2) implies $H^{-n}(\mathbf{N}^\bullet_X)_p$ is also pure of weight zero.

□

From the introduction, we have the inclusions $\hat{m} : \Sigma X \backslash \mathcal{W} \hookrightarrow X$ and $\hat{l} : \mathcal{W} \hookrightarrow X$ which give the distinguished triangle

$$m_* m^! i^* \mathbf{N}^\bullet_X \to i^* \mathbf{N}^\bullet_X \to l_* \hat{l}^* \mathbf{N}^\bullet_X \xrightarrow{+1} .$$

By Lemma 3.2.0.1, Proposition 3.2.0.2, and Proposition 3.2.0.5, we now have a short exact sequence of perverse sheaves underlying a short exact sequence of mixed Hodge modules (Corollary 2.20 [66])

$$0 \to m_* {}^p H^0(m^! i^* \mathbf{N}^\bullet_X) \to i^* \mathbf{N}^\bullet_X \to \mathbf{I}^\bullet_{\Sigma X}(\hat{l}^* \mathbf{N}^\bullet_X) \to 0, \tag{3.3}$$

where $i^* \mathbf{N}^\bullet_X$ has weight $\leq n-1$ (recall $\mathbf{N}^\bullet_X$ has weight $\leq n-1$, and i^* does not increase weights [64] pg. 340), and $\mathbf{I}^\bullet_{\Sigma X}(\hat{l}^* \mathbf{N}^\bullet_X)$ has weight $n-1$. Since a short exact sequence of mixed Hodge modules is strictly compatible with the weight filtration, and the functor Gr^W_{n-1} is exact on the Abelian category of polarizable mixed Hodge modules, we have the short exact sequence of mixed Hodge modules and their underlying perverse sheaves

$$0 \to \mathrm{Gr}^W_{n-1}\, m_* {}^p H^0(m^! i^* \mathbf{N}^\bullet_X) \to \mathrm{Gr}^W_{n-1}\, i^* \mathbf{N}^\bullet_X \to \mathbf{I}^\bullet_{\Sigma X}(\hat{l}^* \mathbf{N}^\bullet_X) \to 0.$$

Theorem 3.2.0.6. *Suppose X is a parameterized space. Then, there is an isomorphism $\operatorname{Gr}^W_{n-1} i^*\mathbf{N}^\bullet_X \cong \mathbf{I}^\bullet_{\Sigma X}(\hat{l}^*\mathbf{N}^\bullet_X)$, so that the short exact sequence of perverse sheaves on X*

$$0 \to m_* {}^pH^0(m^! i^*\mathbf{N}^\bullet_X) \to i^*\mathbf{N}^\bullet_X \to \mathbf{I}^\bullet_{\Sigma X}(\hat{l}^*\mathbf{N}^\bullet_X) \to 0$$

identifies $W_{n-2}\, i^*\mathbf{N}^\bullet_X \cong m_* {}^pH^0(m^! i^*\mathbf{N}^\bullet_X)$.

Proof. Since $\operatorname{Gr}^W_{n-1} i^*\mathbf{N}^\bullet_X$ underlies a pure Hodge module, it is by definition semi-simple as a perverse sheaf, i.e., a direct sum of simple intersection cohomology sheaves with irreducible support. Hence, we can write $\operatorname{Gr}^W_{n-1} i^*\mathbf{N}^\bullet_X$ as direct sum of a semi-simple perverse sheaf $\mathbf{M}^\bullet$ with support in $\Sigma X \backslash \mathcal{W}$ and a semi-simple perverse sheaf whose summands are all not supported on $\Sigma X \backslash \mathcal{W}$. This second semi-simple perverse sheaf has to be $\mathbf{I}^\bullet_{\Sigma X}(\hat{l}^*\mathbf{N}^\bullet_X)$, by pulling back the short exact sequence (3.3) by $\hat{l}^*$.

Finally, we claim $\mathbf{M}^\bullet = 0$. Since $\mathbf{M}^\bullet$ is a direct summand of $\operatorname{Gr}^W_{n-1} i^*\mathbf{N}^\bullet_X$, we have a surjection of perverse sheaves

$$i^*\mathbf{N}^\bullet_X \twoheadrightarrow \operatorname{Gr}^W_{n-1} i^*\mathbf{N}^\bullet_X \twoheadrightarrow \mathbf{M}^\bullet.$$

But ${}^pH^0(m^*)$ is right exact for the perverse t-structure (since m^* is a closed inclusion), so we also get a surjection

$$0 = {}^pH^0(\hat{m}^*\mathbf{N}^\bullet_X) \twoheadrightarrow {}^pH^0(m^*\mathbf{M}^\bullet) = \mathbf{M}^\bullet \to 0,$$

where the last equality follows from the fact that $\mathbf{M}^\bullet$ is supported on $\Sigma X \backslash \mathcal{W}$.

□

Corollary 3.2.0.7. *There are isomorphisms*

$$\operatorname{Gr}^W_{n-1} \mathbb{Q}^\bullet_X[n] \cong \operatorname{Gr}^W_{n-1} \mathbf{N}^\bullet_X \cong i_*\mathbf{I}^\bullet_{\Sigma X}(\hat{l}^*\mathbf{N}^\bullet_X),$$

and

$$W_{n-2}\mathbb{Q}^\bullet_X[n] \cong W_{n-2}\mathbf{N}^\bullet_X \cong \hat{m}_* {}^pH^0(m^! i^*\mathbf{N}^\bullet_X).$$

As mentioned in the introduction, this trivially follows from the fact that i_* preserves weights ([64], pg. 339), is exact for the perverse t-structure, and from the fact that $i_*i^*\mathbf{N}^\bullet_X \cong \mathbf{N}^\bullet_X$, since the support of $\mathbf{N}^\bullet_X$ is contained in ΣX.

At first glance, the formula for $W_{n-2}i^*\mathbf{N}^\bullet_X$ appears quite abstruse. We now give a much more geometric interpretation of this perverse sheaf.

Theorem 3.2.0.8. *Let g be a complex analytic function on ΣX such that $V(g)$ contains $\Sigma X\backslash\mathcal{W}$, but does not contain any irreducible component of ΣX. Then,*

$$W_{n-2}i^*\mathbf{N}^\bullet_X \cong {m'}_* \ker\{\phi_g[-1]i^*\mathbf{N}^\bullet_X \xrightarrow{\text{var}} \psi_g[-1]i^*\mathbf{N}^\bullet_X\},$$

where the kernel is taken in the category of perverse sheaves on ΣX, var *is the variation morphism, and $m' : V(g) \hookrightarrow \Sigma X$ is the closed inclusion.*

Proof. We first note that such a function g exists locally by the prime avoidance lemma. Then, $\Sigma X\backslash V(g) \subseteq \mathcal{W}$, and we have as perverse sheaves

$$\mathbf{I}^\bullet_{\Sigma X}(i^*\mathbf{N}^\bullet_{X|_{\Sigma X\backslash V(g)}}) \cong \mathbf{I}^\bullet_{\Sigma X}(\hat{l}^*\mathbf{N}^\bullet_X),$$

since the normalization is still a covering projection away from $V(g)$ in ΣX. One notes then that the proofs of Proposition 3.2.0.2, Proposition 3.2.0.5, and Theorem 3.2.0.6 remain unchanged with these new choices of complementary subspaces $V(g) \overset{m'}{\hookrightarrow} \Sigma X$ and $\Sigma X\backslash V(g) \overset{l'}{\hookrightarrow} \Sigma X$, so that

$$\mathrm{Gr}^W_{n-1}\, i^*\mathbf{N}^\bullet_X \cong \mathbf{I}^\bullet_{\Sigma X}(i^*\mathbf{N}^\bullet_{X|_{\Sigma X\backslash V(g)}})$$

and

$$W_{n-2}i^*\mathbf{N}^\bullet_X \cong m'_*\,{}^pH^0(m'^!i^*\mathbf{N}^\bullet_X).$$

The claim then follows by taking the long exact sequence in perverse cohomology of the variation distinguished triangle

$$\phi_g[-1]i^*\mathbf{N}^\bullet_X \xrightarrow{\text{var}} \psi_g[-1]i^*\mathbf{N}^\bullet_X \to m'^![1]i^*\mathbf{N}^\bullet_X \xrightarrow{+1},$$

yielding

$$0 \to {}^pH^0(m'^! i^* \mathbf{N}^\bullet_X) \to \phi_g[-1] i^* \mathbf{N}^\bullet_X \xrightarrow{\text{var}} \psi_g[-1] i^* \mathbf{N}^\bullet_X \to {}^pH^1(m'^! i^* \mathbf{N}^\bullet_X) \to 0.$$

□

3.3 The Weight Zero Part in the Surface Case

Suppose $X = V(f)$ is a parameterized surface in $\mathbb{C}^3$; we want to compute $W_0\mathbb{Q}^\bullet_{V(f)}[2]$ using the isomorphism

$$W_0\mathbb{Q}^\bullet_{V(f)}[2] = W_0\mathbf{N}^\bullet_{V(f)} \cong \hat{m}_* {}^pH^0(m^! i^* \mathbf{N}^\bullet_{V(f)}).$$

The main tool we use is the following: if $\dim_{\mathbf{0}} \Sigma f = 1$, then $\Sigma f \backslash \mathcal{W}$ is zero-dimensional (or empty), and perverse cohomology on a zero-dimensional space is just ordinary cohomology. Recall that $\widetilde{V(f)} \xrightarrow{\pi} V(f)$ is the normalization map.

Theorem 3.3.0.1. *Suppose $V(f)$ is a parameterized surface in $\mathbb{C}^3$. Then,*

$$W_0\mathbb{Q}^\bullet_{V(f)}[2] \cong V^\bullet_{\{\mathbf{0}\}}$$

is a perverse sheaf concentrated on a single point, i.e., a finite-dimensional $\mathbb{Q}$-vector space, of dimension

$$\dim V = 1 - |\pi^{-1}(\mathbf{0})| + \sum_C \dim \ker\{\mathrm{id} - h_C\},$$

where $\{C\}$ is the collection of irreducible components of Σf at $\mathbf{0}$, and for each component C, h_C is the (internal) monodromy operator on the local system $H^{-1}(\mathbf{N}^\bullet_{V(f)})_{|C\backslash\{\mathbf{0}\}}$. Note that $|\pi^{-1}(\mathbf{0})|$ is, of course, equal to the number of irreducible components of $V(f)$ at $\mathbf{0}$.

Proof. First, note that we have $\Sigma f \backslash \mathcal{W} = \{\mathbf{0}\}$ (see Example 3.1.0.2), and $\mathcal{W} = \bigcup_C (C \backslash \{\mathbf{0}\})$, where each $C\backslash\{\mathbf{0}\}$ is homeomorphic to a punctured complex disk. Then, we find

$${}^pH^0(m^! i^* \mathbf{N}^\bullet_{V(f)}) \cong H^0(m^! i^* \mathbf{N}^\bullet_{V(f)}) \cong \mathbb{H}^0(\Sigma f, \Sigma f \backslash \{\mathbf{0}\}; i^* \mathbf{N}^\bullet_{V(f)})$$

We can compute this last term from the long exact sequence in relative hypercohomology with coefficients in $\mathbf{N}^\bullet_{V(f)}$:

$$0 \to \mathbb{H}^{-1}(\Sigma f, \Sigma f\backslash\{\mathbf{0}\}; i^*\mathbf{N}^\bullet_{V(f)}) \to H^{-1}(\mathbf{N}^\bullet_{V(f)})_{\mathbf{0}} \to \mathbb{H}^{-1}(\Sigma f\backslash\{\mathbf{0}\}; i^*\mathbf{N}^\bullet_{V(f)}) \to$$
$$\mathbb{H}^{0}(\Sigma f, \Sigma f\backslash\{\mathbf{0}\}; i^*\mathbf{N}^\bullet_{V(f)}) \to H^{0}(\mathbf{N}^\bullet_{V(f)})_{\mathbf{0}} \to \mathbb{H}^{0}(\Sigma f\backslash\{\mathbf{0}\}; i^*\mathbf{N}^\bullet_{V(f)}) \to 0$$

The cosupport condition on $i^*\mathbf{N}^\bullet_{V(f)}$ implies $\mathbb{H}^{-1}(\Sigma f, \Sigma f\backslash\{\mathbf{0}\}; i^*\mathbf{N}^\bullet_{V(f)}) = 0$. Additionally, since $H^0(\mathbf{N}^\bullet_{V(f)})$ is only supported on $\{\mathbf{0}\}$, it follows that $\mathbb{H}^{0}(\Sigma f\backslash\{\mathbf{0}\}; i^*\mathbf{N}^\bullet_{V(f)}) = 0$ as well. Since the normalization of $V(f)$ is rational homology manifold, $H^0(\mathbf{N}^\bullet_{V(f)})_{\mathbf{0}} = 0$ by Theorem 1.1.1.4, and $\dim H^{-1}(\mathbf{N}^\bullet_{V(f)})_{\mathbf{0}} = |\pi^{-1}(\mathbf{0})| - 1$.

Finally,

$$\begin{aligned}\mathbb{H}^{-1}(\Sigma f\backslash\{\mathbf{0}\}; i^*\mathbf{N}^\bullet_{V(f)}) &\cong \mathbb{H}^{-1}(\bigcup_C C\backslash\{\mathbf{0}\}; i^*\mathbf{N}^\bullet_{V(f)}) \\ &\cong \bigoplus_C \mathbb{H}^{-1}(C\backslash\{\mathbf{0}\}; i^*\mathbf{N}^\bullet_{V(f)}).\end{aligned}$$

This last term is easily seen to be (the sum of) global sections of the local system $H^{-1}(\mathbf{N}^\bullet_{V(f)})_{|_{C\backslash\{\mathbf{0}\}}}$, which is just $\ker\{\mathrm{id} - h_C\}$. Taking the alternating sums of the dimensions of the terms in the resulting short exact sequence

$$0 \to H^{-1}(\mathbf{N}^\bullet_{V(f)})_{\mathbf{0}} \to \bigoplus_C \ker\{\mathrm{id} - h_C\} \to \mathbb{H}^{0}(\Sigma f, \Sigma f\backslash\{\mathbf{0}\}; i^*\mathbf{N}^\bullet_{V(f)}) \to 0$$

yields the desired result. □

Example 3.3.0.2. Let $f(x,y,t) = y^2 - x^3 - tx^2$, so that $V(f)$ is the Whitney umbrella. Then, $\Sigma f = V(x,y)$, and $V(f)$ has (smooth) normalization given by $\pi(t,u) = (u^2 - t, u(u^2-t), t)$. Then, it is easy to see that the internal monodromy operator h_C along the component $V(x,y)$ is multiplication by -1, so $\ker\{\mathrm{id} - h_C\} = 0$. Hence,

$$W_0\mathbb{Q}^\bullet_{V(f)}[2] = 0.$$

Example 3.3.0.3. Let $f(x, y, z) = xz^2 - y^3$, so that $\Sigma f = V(y, z)$. Then, the normalization $\widetilde{V(f)}$ is equal to

$$\widetilde{V(f)} = V(u^2 - xy, uy - xz, uz - y^2) \subseteq \mathbb{C}^4,$$

(i.e., the affine cone over the twisted cubic) and the normalization map π is induced by the projection $(u, x, y, z) \mapsto (x, y, z)$. By Subsection 1.1.2, [21], $\widetilde{V(f)}$ is a rational homology manifold. The internal monodromy operator h_C on $H^{-1}(\mathbf{N}^\bullet_{V(f)})_{|V(y,z)\backslash\{\mathbf{0}\}}$ is trivial, so $\ker\{\mathrm{id} - h_C\} \cong \mathbb{Q}$. Thus,

$$W_0\mathbb{Q}^\bullet_{V(f)}[2] \cong \mathbb{Q}^\bullet_{\{\mathbf{0}\}}.$$

Example 3.3.0.4. $f(x, y, z) = xyz$, so $\Sigma f = V(x, y) \cup V(y, z) \cup V(x, z)$. Then, $|\pi^{-1}(\mathbf{0})| = 3$, and the internal monodromy operators h_C are all the identity. It then follows that

$$W_0\mathbb{Q}^\bullet_{V(f)}[2] \cong \mathbb{Q}^\bullet_{\{\mathbf{0}\}}.$$

Remark 3.3.0.5. It is an interesting question whether or not $\dim_{\mathbf{0}} H^0(W_0\mathbb{Q}^\bullet_{V(f)}[2])_{\mathbf{0}}$ is some sort of invariant of the surface $V(f)$.

One would hope that this dimension is an invariant of the local, ambient topological type of $V(f)$ at the origin, but this is distinctly non-obvious–this perverse sheaf is defined in terms of the normalization of $V(f)$ and the internal monodromy of $H^{-1}(\mathbf{N}^\bullet_{V(f)})$. While the normalization of $V(f)$ is unique, it is an analytic map, and need not be preserved under local homemorphisms.

Remark 3.3.0.6. It is an interesting question to ask when $W_0\mathbb{Q}^\bullet_{V(f)}[2] = V^\bullet_{\{\mathbf{0}\}} = 0$. The most natural candidates to examine are those parameterized surfaces with smooth singular loci. We have seen above with the Whitney umbrella that it is possible for V to vanish when the critical locus is smooth; However, we know smoothness alone is not sufficient, as one sees with $V(xz^2 - y^3)$.

One can, however, distinguish these examples by noting that the normalization of the Whitney umbrella is smooth, and that of the surface $V(xz^2-y^3)$ is not. Does this comparison hold in general? The normalization of the Whitney umbrella can also be realized as a simultaneous normalization of the family $V(y^2-x^3-t_0x^2, t-t_0)$ of plane curve singularities, while the normalization of $V(xz^2-y^3)$ does not admit such a description.

This would make the perverse sheaf $W_0\mathbb{Q}^\bullet_{V(f)}[2]$ very relevant to **Lê's Conjecture** (see Conjecture 4.0.0.1).

3.3.1 Interpretation via Invariant Jordan Blocks of the Monodromy

Let $V(f) \subseteq \mathbb{C}^3$ be a parameterized surface, and let L be an IPA-form for f at $\mathbf{0}$. Then, $f_\xi := f_{|_{V(L-\xi)}}$ defines a plane curve singularity inside $V(L-\xi)$ for $|\xi| \ll 1$, and it is well-known that the stalks of the intersection cohomology complex $\mathbf{I}^\bullet_{V(f_\xi)} \cong \mathbf{I}^\bullet_{V(f)_{|_{V(L-\xi)}}}[-1]$ at $p \in \Sigma f_\xi$ satisfy

$$H^j(\mathbf{I}^\bullet_{V(f_\xi)})_p \cong \begin{cases} \mathbb{Q}^{J_1(f_\xi)_p+1} & \text{if } j = 0 \\ 0 & \text{if } j \neq 0 \end{cases}$$

where $J_1(f_\xi)_p$ denotes the number of Jordan blocks of the eigenvalue 1 for the Milnor monodromy action on the cohomology group $H^1(F_{f_\xi,p};\mathbb{Q})$ (see e.g., Chapter 8 of [55], or Lemma 4.3 [59]). It is then immediate from the fundamental short exact sequence that $\dim H^0(\mathbf{N}^\bullet_{V(f_\xi)})_p = J_1(f_\xi)_p$ inside each curve $V(f_\xi)$.

Thus, for the total surface $V(f)$ we have $\dim H^{-1}(\mathbf{N}^\bullet_{V(f)})_\mathbf{0} = J_1(f_0)_\mathbf{0}$, and along an irreducible component C of Σf, the stalks $H^{-1}(\mathbf{N}^\bullet_{V(f)})_p$ have generic dimension $J_1(f_\xi)_C := J_1(f_\xi)_p$. We can then identify the internal monodromy operator h_C on $H^{-1}(\mathbf{N}^\bullet_{V(f)})_p$ as acting on the collection of Jordan blocks for the generic *transversal* Milnor monodromy. Hence, we can interpret $\ker\{\mathrm{id} - h_C\}$ as those Jordan blocks of the transversal Milnor monodromy that are *invariant* under the internal monodromy around a component C of Σf.

3.4 Connection with the Vanishing Cycles

In [39], Massey shows that, for an arbitrary (reduced) hypersurface $V(f)$ in some open neighborhood $\mathcal{U}$ of the origin in $\mathbb{C}^{n+1}$, one has a isomorphism of **perverse sheaves** $\mathbf{N}^\bullet_{V(f)} \cong \ker\{\mathrm{id} - \widetilde{T}_f\}$, where $\widetilde{T}_f$ is the Milnor monodromy action on the vanishing cycles $\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]$ (this isomorphism holds for $\mathbb{Q}$ coefficients, where one may also obtain this result using the language of mixed Hodge modules).

However, $\mathrm{id} - \widetilde{T}_f$ is not a morphism of mixed Hodge modules; to correct this, one instead considers the morphism $N = \frac{1}{2\pi i}\log T_u$, where T_u is the unipotent part of the monodromy operator $\widetilde{T}_f$. In this case, $\ker\{\mathrm{id} - \widetilde{T}_f\} \cong \ker N$ as perverse sheaves, and we consider $\ker N$ as a subobject of the unipotent vanishing cycles $\phi_{f,1}[-1]\mathbb{Q}^\bullet_{\mathcal{U}}[n+1]$.

On the level of mixed Hodge modules, we have an isomorphism

$$\mathbf{N}^\bullet_{V(f)} \cong \ker N(1)$$

where (1) denotes the Tate twist operation. This description follows from Massey's original proof for perverse sheaves [39], with the following changes. Let $j : V(f) \hookrightarrow \mathcal{U}$. Starting from the two short exact sequences of mixed Hodge modules

$$0 \to j^*[-1]\mathbb{Q}^\bullet_{\mathcal{U}}[n+1] \to \psi_{f,1}[-1]\mathbb{Q}^\bullet_{\mathcal{U}}[n+1] \overset{\mathrm{can}}{\to} \phi_{f,1}[-1]\mathbb{Q}^\bullet_{\mathcal{U}}[n+1] \to 0 \tag{3.4}$$

and

$$0 \to \phi_{f,1}[-1]\mathbb{Q}^\bullet_{\mathcal{U}}[n+1] \overset{\mathrm{var}}{\to} \psi_{f,1}[-1]\mathbb{Q}^\bullet_{\mathcal{U}}[n+1](-1) \to j^![1]\mathbb{Q}^\bullet_{\mathcal{U}}[n+1] \to 0,$$

(note the variation morphism now has a Tate twist of (-1)), so that $N = \mathrm{can} \circ \mathrm{var}$. Then, if $i : \Sigma f \hookrightarrow V(f)$, we obtain the isomorphism

$$\phi_{f,1}[-1]\mathbb{Q}^\bullet_{\mathcal{U}}[n+1](1) \overset{i_* {}^pH^0(i^! \mathrm{var})}{\longrightarrow} i_* {}^pH^0(i^! \psi_{f,1}[-1]\mathbb{Q}^\bullet_{\mathcal{U}}[n+1])$$

since $(j \circ i)^![1]\mathbb{Q}^\bullet_{\mathcal{U}}[n+1] \in {}^pD^{\geq 0}(V(f) \backslash \Sigma f)$. This, together with the isomorphisms

$$\mathbf{N}^\bullet_{V(f)} \cong i_* {}^pH^0(i^! j^*[-1]\mathbb{Q}^\bullet_{\mathcal{U}}[n+1]) \cong i_* {}^pH^0(i^! \ker \mathrm{can})$$

obtained by Lemma 3.2.0.1 and applying $i_* {}^p H^0(i^!)$ to (3.4) yields the final identification

$$\mathbf{N}^\bullet_{V(f)} \cong \ker N(1).$$

Hence, $W_k \mathbf{N}^\bullet_{V(f)} \cong W_k \ker N(1) \cong W_{k+2} \ker N$ for all $k \leq n-1$.

Remark 3.4.0.1. Massey's result $\mathbf{N}^\bullet_{V(f)} \cong \ker\{\mathrm{id} - \widetilde{T}_f\}$ holds for arbitrary reduced hypersurfaces, while the complex $\mathbf{N}^\bullet_X$ exists via the fundamental short exact sequence whenever the complex $\mathbb{Q}^\bullet_X[n]$ is perverse (and X is reduced and purely n-dimensional).

It is a very interesting open question if this (or a similar) interpretation of $\mathbf{N}^\bullet_X$ exists for this more general case (for perverse sheaves or for mixed Hodge modules). For example, if X is a local complete intersection, can one interpret $\mathbf{N}^\bullet_X$ in terms of monodromies of the defining functions of X?

3.5 The Algebraic Setting and Saito's Work

Morihiko Saito has recently drawn interesting connections with the comparison complex $\mathbf{N}^\bullet_X$ in the setting of (algebraic) mixed Hodge modules in a recent preprint [67]. In particular, Saito shows, for an arbitrary reduced complex algebraic variety X of pure dimension n, that the weight zero part of the cohomology group $H^1(X;\mathbb{Q})$ is given by

$$W_0 H^1(X;\mathbb{Q}) \cong \operatorname{coker}\{H^0(\widetilde{X};\mathbb{Q}) \to H^0(X;\mathcal{F}_X)\},$$

where $\pi : \widetilde{X} \to X$ is the normalization of X, and $\mathcal{F}_X$ is a certain constructible sheaf on X, given by the cokernel of the natural morphism of sheaves $\mathbb{Q}_X \to \pi_*\mathbb{Q}_{\widetilde{X}}$. The algebraic setting is necessary here, in order to endow $H^0(X;\mathcal{F}_X)$ with a mixed Hodge structure, and for working in the derived category of mixed Hodge modules.

This constructible sheaf $\mathcal{F}_X$ is none other than the cohomology sheaf $H^{-n+1}(\mathbf{N}^\bullet_X)$; this follows immediately from taking the long exact sequence in cohomology of the fundamental short exact sequence of the normalization. If, as in Saito's case, the sheaf $\mathbb{Q}^\bullet_X[n]$ is not perverse, one can obtain this isomorphism from the distinguished triangle

$$\mathcal{F}_X[n] \to \mathbf{N}^\bullet_X[1] \to \pi_*\mathbf{N}^\bullet_{\widetilde{X}}[1] \xrightarrow{+1} \tag{3.5}$$

obtained via the octahedral axiom in the derived category $D^b_c(X)$, together with the fact that $\widetilde{X}$ is normal. Indeed, since $H^i(\mathbf{N}^\bullet_X) = 0$ for $i < -n+1$, and $H^i(\mathbf{N}^\bullet_{\widetilde{X}}) = 0$ for $i < -n+2$ (as $\widetilde{X}$ is locally irreducible, Lemma 1.1.1.5 implies $H^{-n+1}(\mathbf{N}^\bullet_{\widetilde{X}}) = 0$), the isomorphism $\mathbf{F}^\bullet_X \xrightarrow{\sim} H^{-n+1}(\mathbf{N}^\bullet_X)$ follows from the long exact sequence in stalk cohomology applied to (3.5).

Consequently, we can interpret Saito's result as an isomorphism

$$W_0H^1(X;\mathbb{Q}) \cong \operatorname{coker}\{H^0(\widetilde{X};\mathbb{Q}) \to \mathbb{H}^{-n+1}(X;\mathbf{N}^\bullet_X)\},$$

since $H^0(X; H^{-n+1}(\mathbf{N}^\bullet_X)) \cong \mathbb{H}^{-n+1}(X;\mathbf{N}^\bullet_X)$.

It would seem to be an interesting question in the local analytic case to relate this result with the isomorphism $\mathbf{N}^\bullet_X \cong W_{n-1}\mathbb{Q}^\bullet_X[n]$, and results obtained in Theorem 3.2.0.6 and Corollary 3.2.0.7 where $\mathbf{N}^\bullet_X$ is endowed with the natural structure of a mixed Hodge module on X (instead of as a complex of mixed Hodge modules in the algebraic context in Saito's work).

Chapter 4

Bobadilla's Conjecture and the Beta Invariant

Our work on the singularities of parameterized spaces led us to a well-known conjecture of Lê Dũng Tráng [62],[8] regarding the equisingularity of parameterized surface singularities.

Conjecture 4.0.0.1 (Lê, D.T.). *Suppose* $(V(f), \mathbf{0}) \subseteq (\mathbb{C}^3, \mathbf{0})$ *is a reduced hypersurface with* $\dim_{\mathbf{0}} \Sigma f = 1$*, for which the normalization of* $V(f)$ *is a bijection. Then, in fact,* $V(f)$ *is the total space of an equisingular deformation of plane curve singularities.*

Lê's Conjecture is a generalization of Mumford's Theorem [65] stating that, if the real link of a normal surface singularity is a topological sphere, then the singularity is in fact smooth. Although Lê's conjecture has been shown to be true for some special cases (e.g. for cyclic covers over normal surface singularities totally ramified along the zero locus of an analytic function by Luengo and Pichon [34], and for surface singularities containing a smooth curve through the origin by Bobadilla [8]), no general proof or counterexample is known.

Bobadilla's work on Lê's Conjecture led him to a related problem regarding the equsingularity of arbitrary hypersurfaces with one-dimensional singular loci, which we will refer to as Bobadilla's Conjecture (Conjecture 4.1.0.2) and later, as the Beta Conjecture (Conjecture 4.1.0.5) and will be the focus of this chapter. Using the **beta invariant**, β_f, of a function f with one-dimensional critical locus (Definition 4.1.0.3), the author and David Massey prove in [24] some special cases of Bobadilla's Conjecture; these are the first known positive results toward the conjecture. In this chapter, we prove Bobadilla's Conjecture in two special cases:

1. In Corollary 4.2.0.2, we prove an induction-like result for when f is a sum of two analytic functions defined on disjoint sets of variables.

2. In Theorem 4.3.0.2, we prove the result for the case when the relative polar curve Γ^1_{f,z_0} is defined by a single equation inside the relative polar surface $\Gamma^2_{f,\mathbf{z}}$ (see below).

4.1 Bobadilla's Conjecture

Suppose that $\mathcal{U}$ is an open neighborhood of the origin in $\mathbb{C}^{n+1}$, and that $f : (\mathcal{U}, \mathbf{0}) \to (\mathbb{C}, 0)$ is a complex analytic function with a 1-dimensional critical locus at the origin, i.e., $\dim_{\mathbf{0}} \Sigma f = 1$. We use coordinates $\mathbf{z} := (z_0, \cdots, z_n)$ on $\mathcal{U}$.

We assume that z_0 is an IPA form for f at $\mathbf{0}$ so that $\dim_{\mathbf{0}} \Sigma(f_{|_{V(z_0)}}) = 0$. One implication of this is that

$$V\left(\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \ldots, \frac{\partial f}{\partial z_n}\right)$$

is purely 1-dimensional at the origin. As analytic cycles, we write

$$\left[V\left(\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \ldots, \frac{\partial f}{\partial z_n}\right)\right] \;=\; \Gamma^1_{f,z_0} + \Lambda^1_{f,z_0},$$

where Γ^1_{f,z_0} and Λ^1_{f,z_0} are, respectively, the relative polar curve and 1-dimensional Lê cycle of f with respect to z_0; see [42] or Appendix A.

We recall a classical non-splitting result (presented in a convenient form here) proved independently by Gabrielov, Lazzeri, and Lê (in [10], [26], and [30], respectively) regarding the non-splitting of the cohomology of the Milnor fiber of $f_{|_{V(z_0)}}$ over the critical points of f in a nearby hyperplane slice $V(z_0 - t)$ for a small non-zero value of t.

Theorem 4.1.0.1 (GLL non-splitting)**.** *The following are equivalent:*

1. *The Milnor number of $f_{|_{V(z_0)}}$ at the origin is equal to*

$$\sum_C \mu_C^\circ \left(C \cdot V(z_0)\right)_{\mathbf{0}},$$

where the sum is over the irreducible components C of Σf at $\mathbf{0}$, $(C \cdot V(z_0))_{\mathbf{0}}$ denotes the intersection number of C and $V(z_0)$ at $\mathbf{0}$, and μ_C° denotes the Milnor number of f, restricted to a generic hyperplane slice, at a point $\boldsymbol{p} \in C \backslash \{\mathbf{0}\}$ close to $\mathbf{0}$.

2. *Γ^1_{f,z_0} is zero at the origin (i.e., $\mathbf{0}$ is not in the relative polar curve).*

Furthermore, when these equivalent conditions hold, Σf has a single irreducible component which is smooth and is transversely intersected by $V(z_0)$ at the origin.

This chapter is concerned with a recent conjecture made by Javier Fernández de Bobadilla, positing that, in the spirit of Theorem 4.1.0.1, the cohomology of the Milnor fiber of f, **not of a hyperplane slice**, does not split. We state a slightly more general form of Bobadilla's original conjecture, for the case where Σf may, a priori, have more than a single irreducible component:

Conjecture 4.1.0.2 (Fernández de Bobadilla)**.** *Suppose that $\widetilde{H}^k(F_{f,\mathbf{0}};\mathbb{Z})$ is non-zero only in degree $(n-1)$, and that*

$$\widetilde{H}^{n-1}(F_{f,\mathbf{0}};\mathbb{Z}) \cong \bigoplus_C \mathbb{Z}^{\mu_C^\circ},$$

where the sum is over all irreducible components C of Σf at $\mathbf{0}$. Then, in fact, Σf has a single irreducible component, which is smooth.

Bobadilla's conjecture first appeared in [3] as a series of three conjectures A, B, and C, although we most directly address Conjecture C in our phrasing of Conjecture 4.1.0.2 (see the introduction of [3]).

We approach Bobadilla's Conjecture via the **beta invariant** of a hypersurface with a 1-dimensional critical locus, first defined and explored Massey in [36]. The beta invariant, β_f, of f is an invariant of the local ambient topological-type of the hypersurface $V(f)$. It is a non-negative integer, and is algebraically calculable.

Our motivation for using this invariant is that the requirement that $\beta_f = 0$ is precisely equivalent to the hypotheses of Conjecture 4.1.0.2, essentially turning the problem into a purely algebraic question (see Theorem 5.4 of [36]). For this reason, we will refer to our new formulation of Conjecture 4.1.0.2 as the **Beta Conjecture**.

Recall that, since $\dim_{\mathbf{0}} \Sigma\left(f_{|_{V(z_0)}}\right) \leq 0$, the Lê numbers $\lambda^1_{f,z_0}(\mathbf{0})$ and $\lambda^1_{f,\mathbf{z}}(\mathbf{0})$ are defined.

Definition 4.1.0.3 (Definition 3.1, [36])**.** *The* ***beta invariant*** *of f, denoted β_f, is the number*

$$\begin{aligned}\beta_f &= \lambda^0_{f,z_0}(\mathbf{0}) - \lambda^1_{f,\mathbf{z}}(\mathbf{0}) + \sum_C \mathring{\mu}_C \\ &= \widetilde{b}_n(F_{f,\mathbf{0}}) - \widetilde{b}_{n-1}(F_{f,\mathbf{0}}) + \sum_C \mathring{\mu}_C \\ &= \left(\Gamma^1_{f,z_0} \cdot V(f)\right)_{\mathbf{0}} - \mu_{\mathbf{0}}\left(f_{|_{V(z_0)}}\right) + \sum_C \mathring{\mu}_C,\end{aligned}$$

where C runs over the collection of irreducible components of Σf at $\mathbf{0}$, and $\mathring{\mu}_C$ denotes the generic transversal Milnor number of f along C, and $\widetilde{b}_i(F_{f,\mathbf{0}})$ denotes the Betti number of the i-th reduced cohomology group $\widetilde{H}^i(F_{f,\mathbf{0}};\mathbb{Z})$.

Remark 4.1.0.4. A key property of the beta invariant is that the value β_f is independent of the choice of linear form z_0 (provided, of course, that the linear form satisfies $\dim_{\mathbf{0}} \Sigma(f_{|_{V(z_0)}}) = 0$). This often allows a great deal of freedom in calculating β_f for a given f, as different choices of linear forms $L = z_0$ may result in simpler expressions for the intersection numbers λ^0_{f,z_0} and λ^1_{f,z_0}, while leaving the value of β_f unchanged. [See 36, Remark 3.2, Example 3.4].

It is shown in [36] that $\beta_f \geq 0$. The interesting question is how strong the requirement that $\beta_f = 0$ is.

Conjecture 4.1.0.5 (Beta Conjecture)**.** *If $\beta_f = 0$, then Σf has a single irreducible component at $\mathbf{0}$, which is smooth.*

Conjecture 4.1.0.6 (Polar Form of the Beta Conjecture)**.** *If $\beta_f = 0$, then $\mathbf{0}$ is not in the relative polar curve Γ^1_{f,z_0} (i.e., the relative polar curve is 0 as a cycle at the origin).*

Equivalently, if the relative polar curve at the origin is not empty, then $\beta_f > 0$.

Proposition 4.1.0.7. *The Beta Conjecture is equivalent to the Polar Form of the Beta Conjecture.*

Proof. Suppose throughout that $\beta_f = 0$.

Suppose first that the Beta Conjecture holds, so that Σf has a single irreducible component at $\mathbf{0}$, which is smooth. Then $\beta_f = \lambda^0_{f,z_0} = 0$, and so the relative polar curve must be zero at the origin.

Suppose now that the polar form of the Beta Conjecture holds, so that $\Gamma^1_{f,z_0} = 0$ at $\mathbf{0}$. Then GLL Non-Splitting implies that Σf has a single irreducible component at $\mathbf{0}$, which is smooth. $\square$

We will need the following well-known results regarding intersection numbers later on in this chapter.

Proposition 4.1.0.8. *Suppose f is an IPA-deformation of $f_{|_{V(z_0)}}$ at $\mathbf{0}$. Then,* *1.* $\dim_{\mathbf{0}} \Gamma^1_{f,z_0} \cap V\left(\frac{\partial f}{\partial z_0}\right) \leq 0$*, and*

$$\left(\Gamma^1_{f,z_0} \cdot V(f)\right)_{\mathbf{0}} = \left(\Gamma^1_{f,z_0} \cdot V(z_0)\right)_{\mathbf{0}} + \left(\Gamma^1_{f,z_0} \cdot V\left(\frac{\partial f}{\partial z_0}\right)\right)_{\mathbf{0}}.$$

The proof of this result is sometimes referred to as Teissier's trick*.*

2. In addition,

$$\mu_{\mathbf{0}}\left(f_{|_{V(z_0)}}\right) = \left(\Gamma^1_{f,z_0} \cdot V(z_0)\right)_{\mathbf{0}} + \left(\Lambda^1_{f,z_0} \cdot V(z_0)\right)_{\mathbf{0}}.$$

4.2 Generalized Suspension

Suppose that $\mathcal{U}$ and $\mathcal{W}$ are open neighborhoods of the origin in $\mathbb{C}^{n+1}$ and $\mathbb{C}^{m+1}$, respectively, and let $g : (\mathcal{U}, \mathbf{0}) \to (\mathbb{C}, 0)$ and $h : (\mathcal{W}, \mathbf{0}) \to (\mathbb{C}, 0)$ be two complex analytic functions. Let $\pi_1 : \mathcal{U} \times \mathcal{W} \to \mathcal{U}$ and $\pi_2 : \mathcal{U} \times \mathcal{W} \to \mathcal{W}$ be the natural projection maps, and set $f = g \boxplus h := g \circ \pi_1 + h \circ \pi_2$. Then, one trivially has

$$\Sigma f = \left(\Sigma g \times \mathbb{C}^{m+1}\right) \cap \left(\mathbb{C}^{n+1} \times \Sigma h\right).$$

Consequently, if we assume that g has a one-dimensional critical locus at the origin, and that h has an isolated critical point at $\mathbf{0}$, then $\Sigma f = \Sigma g \times \{\mathbf{0}\}$ is 1-dimensional and (analytically) isomorphic to Σg.

From this, one immediately has the following result.

Proposition 4.2.0.1. *Suppose that g and h are as above, so that $f = g \boxplus h$ has a one-dimensional critical locus at the origin in $\mathbb{C}^{n+m+2}$. Then, $\beta_f = \mu_{\mathbf{0}}(h)\beta_g$.*

Proof. This is a consequence of the Sebastiani-Thom isomorphism (see the results of Némethi [60],[61], Oka [63], Sakamoto [68], Sebastiani-Thom [69], and Massey [50]) for the reduced integral cohomology of the Milnor fiber of $f = g \boxplus h$ at $\mathbf{0}$. Letting $\widehat{C}$ denote the component of the critical locus f which corresponds to C, the Sebastiani-Thom Theorem tells us that

$$\widetilde{b}_{n+m+1}(F_{f,\mathbf{0}}) = \mu_{\mathbf{0}}(h)\widetilde{b}_n(F_{g,\mathbf{0}}), \quad \widetilde{b}_{n+m}(F_{f,\mathbf{0}}) = \mu_{\mathbf{0}}(h)\widetilde{b}_{n-1}(F_{g,\mathbf{0}}), \quad \text{and} \quad \mu^{\circ}_{\widehat{C}} = \mu_{\mathbf{0}}(h)\mu^{\circ}_{C}.$$

Thus,

$$\beta_f = \lambda^0_{f,z_0} - \lambda^1_{f,z_0} + \sum_{\widehat{C}} \mu^\circ_{\widehat{C}} = \widetilde{b}_{n+m+1}(F_{f,\mathbf{0}}) - \widetilde{b}_{n+m}(F_{f,\mathbf{0}}) + \sum_{\widehat{C}} \mu^\circ_{\widehat{C}} = \mu_{\mathbf{0}}(h)\beta_g.$$

□

Corollary 4.2.0.2. *Suppose $f = g \boxplus h$, where g and h are as in Proposition 4.2.0.1. Then, the Beta Conjecture is true is true for g if and only if it is true for f.*

Proof. Suppose that $\beta_f = 0$. By Proposition 4.2.0.1, this is equivalent to $\beta_g = 0$, since $\mu_{\mathbf{0}}(h) > 0$. By assumption, $\beta_g = 0$ implies that Σg is smooth at zero. Since $\Sigma f = \Sigma g \times \{\mathbf{0}\}$, it follows that Σf is also smooth at $\mathbf{0}$, i.e., the Beta Conjecture is true for f. The exact same proof then implies the converse. □

4.3 Γ^1_{f,z_0} as a hypersurface in $\Gamma^2_{f,\mathbf{z}}$

Let $I := \langle \frac{\partial f}{\partial z_2}, \cdots, \frac{\partial f}{\partial z_n} \rangle \subseteq \mathcal{O}_{\mathcal{U},\mathbf{0}}$, so that the relative polar surface of f with respect to the coordinates $\mathbf{z}$ is (as a cycle at $\mathbf{0}$) given by $\Gamma^2_{f,\mathbf{z}} = [V(I)]$.

For the remainder of this section, we will drop the brackets around cycles for convenience, and assume that everything is considered as a cycle unless otherwise specified. We remind the reader that we are assuming that $f_{|_{V(z_0)}}$ has an isolated critical point at the origin.

Proposition 4.3.0.1. *The following are equivalent:*

1. $\dim_{\mathbf{0}} \left(\Gamma^2_{f,\boldsymbol{z}} \cap V(f) \cap V(z_0) \right) = 0.$

2. *For all irreducible components C at the origin of the analytic set $\Gamma^2_{f,\boldsymbol{z}} \cap V(f)$, C is purely 1-dimensional and properly intersected by $V(z_0)$ at the origin.*

3. *$\Gamma^2_{f,\boldsymbol{z}}$ is properly intersected by $V(z_0, z_1)$ at the origin.*

Furthermore, when these equivalent conditions hold

$$\left(\Gamma^2_{f,\mathbf{z}} \cdot V(f) \cdot V(z_0)\right)_{\mathbf{0}} \;=\; \mu_{\mathbf{0}}\left(f_{|_{V(z_0)}}\right) + \left(\Gamma^2_{f,\mathbf{z}} \cdot V(z_0, z_1)\right)_{\mathbf{0}}.$$

Proof. Clearly (1) and (2) are equivalent. We wish to show that (1) and (3) are equivalent. This follows from Tessier's trick applied to $f_{|_{V(z_0)}}$, but – as it is crucial – we shall quickly run through the argument.

Since $f_{|_{V(z_0)}}$ has an isolated critical point at the origin,

$$\dim_{\mathbf{0}}\left(\Gamma^2_{f,\mathbf{z}} \cap V\left(\frac{\partial f}{\partial z_1}\right) \cap V(z_0)\right) = 0.$$

Hence, $Z := \Gamma^2_{f,\mathbf{z}} \cap V(z_0)$ is purely 1-dimensional at the origin.

Let Y be an irreducible component of Z through the origin, and let $\alpha(t)$ be a parametrization of Y such that $\alpha(0) = \mathbf{0}$. Let $z_1(t)$ denote the z_1 component of $\alpha(t)$. Then,

$$\big(f(\alpha(t))\big)' \;=\; \frac{\partial f}{\partial z_1}\Big|_{\alpha(t)} \cdot z_1'(t). \tag{$\dagger$}$$

Since $\dim_{\mathbf{0}} Y \cap V\left(\frac{\partial f}{\partial z_1}\right) = 0$, we conclude that $\big(f(\alpha(t))\big)' \equiv 0$ if and only if $z_1'(t) \equiv 0$, which tells us that $f(\alpha(t)) \equiv 0$ if and only if $z_1(t) \equiv 0$. Thus, $\dim_{\mathbf{0}} Y \cap V(f) = 0$ if and only if $\dim_{\mathbf{0}} Y \cap V(z_1) = 0$, i.e., (1) and (3) are equivalent. The equality now follows at once by considering the t-multiplicity of both sides of ($\dagger$). $\square$

Theorem 4.3.0.2. *Suppose that*

1. *for all irreducible components C at the origin of the analytic set $\Gamma^2_{f,\boldsymbol{z}} \cap V(f)$, C is purely 1-dimensional, properly intersected by $V(z_0)$ at the origin, and $(C \cdot V(z_0))_{\mathbf{0}} = \operatorname{mult}_{\mathbf{0}} C$, and*

2. *the cycle Γ^1_{f,z_0} equals $\Gamma^2_{f,\boldsymbol{z}} \cdot V(h)$ for some $h \in \mathcal{O}_{\mathcal{U},\mathbf{0}}$ (in particular, the relative polar curve at the origin is non-empty).*

Then,

$$\widetilde{b}_n(F_{f,\mathbf{0}}) - \widetilde{b}_{n-1}(F_{f,\mathbf{0}}) \geq \left(\Gamma^2_{f,\boldsymbol{z}} \cdot V(z_0, z_1)\right)_{\mathbf{0}}$$

and so

$$\beta_f \geq \left(\Gamma^2_{f,\boldsymbol{z}} \cdot V(z_0, z_1)\right)_{\mathbf{0}} + \sum_C \mu^\circ_C.$$

In particular, the Beta Conjecture is true for f.

Proof. By Proposition 4.3.0.1,

$$\left(\Gamma^2_{f,\mathbf{z}} \cdot V(f) \cdot V(z_0)\right)_{\mathbf{0}} \;=\; \mu_{\mathbf{0}}\left(f_{|_{V(z_0)}}\right) + \left(\Gamma^2_{f,\mathbf{z}} \cdot V(z_0, z_1)\right)_{\mathbf{0}}.$$

By assumption, $\Gamma^1_{f,z_0} = \Gamma^2_{f,\mathbf{z}} \cdot V(h)$, for some $h \in \mathcal{O}_{\mathcal{U},\mathbf{0}}$. Then, via Proposition 4.1.0.8 and the above paragraph, we have

$$\begin{aligned}
\widetilde{b}_n(F_{f,\mathbf{0}}) - \widetilde{b}_{n-1}(F_{f,\mathbf{0}}) &= \lambda^0_{f,z_0} - \lambda^1_{f,z_0} \\
&= \left(\Gamma^1_{f,z_0} \cdot V(f)\right)_{\mathbf{0}} - \mu_{\mathbf{0}}\left(f_{|_{V(z_0)}}\right) \\
&= \left[\left(\Gamma^2_{f,\mathbf{z}} \cdot V(h) \cdot V(f)\right)_{\mathbf{0}} - \left(\Gamma^2_{f,\mathbf{z}} \cdot V(f) \cdot V(z_0)\right)_{\mathbf{0}}\right] + \left(\Gamma^2_{f,\mathbf{z}} \cdot V(z_0, z_1)\right)_{\mathbf{0}}.
\end{aligned}$$

As $(C \cdot V(z_0))_{\mathbf{0}} = \operatorname{mult}_{\mathbf{0}} C$ for all irreducible components C of $\Gamma^2_{f,\mathbf{z}} \cap V(f)$, the bracketed quantity above is non-negative. The conclusion follows. $\square$

Example 4.3.0.3. To illustrate the content of Theorem 4.3.0.2, consider the following example. Let $f = (x^3 + y^2 + z^5)z$ on $\mathbb{C}^3$, with coordinate ordering (x, y, z). Then, we have $\Sigma f = V(x^3 + y^2, z)$, and

$$\Gamma^2_{f,(x,y)} = V\left(\frac{\partial f}{\partial z}\right) = V(x^3 + y^2 + 6z^5),$$

which we note has an isolated singularity at $\mathbf{0}$.

Then,

$$\begin{aligned} V\left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) &= V(2yz, x^3 + y^2 + 6z^5) \\ &= V(y, x^3 + 6z^5) + V(z, x^3 + y^2) \end{aligned}$$

so that $\Gamma^1_{f,x} = V(y, x^3 + 6z^5)$, and $\Lambda^1_{f,x}$ consists of the single component $C = V(z, x^3 + y^2)$ with $\overset{\circ}{\mu}_C = 1$. It is then immediate that

$$\Gamma^1_{f,x} = V(y) \cdot \Gamma^2_{f,(x,y)},$$

so that the second hypothesis of Theorem 4.3.0.2 is satisfied. For the first hypothesis, we note that

$$\begin{aligned} \Gamma^2_{f,(x,y)} \cap V(f) &= V(x^3 + y^2 + 6z^5, (x^3 + y^2 + z^5)z) \\ &= V(5z^5, x^3 + y^2 + z^5) \cup V(x^3 + y^2, z) \\ &= V(x^3 + y^2, z) = C. \end{aligned}$$

Clearly, C is purely 1-dimensional, and is properly intersected by $V(x)$ at $\mathbf{0}$. Finally, we see that

$$(C \cdot V(x))_{\mathbf{0}} = V(x, z, x^3 + y^2)_{\mathbf{0}} = 2 = \operatorname{mult}_{\mathbf{0}} C,$$

so the two hypotheses of Theorem 4.3.0.2 are satisfied.

By Definition 4.1.0.3, Theorem 4.3.0.2 guarantees that the following inequality holds:

$$\lambda^0_{f,x} - \lambda^1_{f,x} \geq \left(\Gamma^2_{f,(x,y)} \cdot V(x, y)\right)_{\mathbf{0}}.$$

Let us verify this inequality ourselves. We have

$$\begin{aligned}\lambda^0_{f,x} &= \left(\Gamma^1_{f,x} \cdot V\left(\frac{\partial f}{\partial x}\right)\right)_{\mathbf{0}} = V(y, x^3 + 6z^5, 3x^2 z)_{\mathbf{0}} \\ &= V(y, x^2, z^5)_{\mathbf{0}} + V(y, z, x^3)_{\mathbf{0}} = 13,\end{aligned}$$

and

$$\lambda^1_{f,x} = \left(\Lambda^1_{f,x} \cdot V(x)\right)_{\mathbf{0}} = V(x, z, x^3 + y^2)_{\mathbf{0}} = 2.$$

Finally, we compute

$$\left(\Gamma^2_{f,(x,y)} \cdot V(x,y)\right)_{\mathbf{0}} = V(x, y, x^3 + y^2 + 6z^5)_{\mathbf{0}} = 5.$$

Putting this all together, we have

$$\lambda^0_{f,x} - \lambda^1_{f,x} = 11 \geq 5 = \left(\Gamma^2_{f,(x,y)} \cdot V(x,y)\right)_{\mathbf{0}},$$

as expected.

Example 4.3.0.4. We now give an example where the relative polar curve is **not** defined inside $\Gamma^2_{f,\mathbf{z}}$ by a single equation, and $\widetilde{b}_n(F_{f,\mathbf{0}}) - \widetilde{b}_{n-1}(F_{f,\mathbf{0}}) < 0$.

Let $f = (z^2 - x^2 - y^2)(z - x)$, with coordinate ordering (x, y, z). Then, we have $\Sigma f = V(y, z - x)$, and

$$\Gamma^2_{f,\mathbf{z}} = V\left(\frac{\partial f}{\partial z}\right) = V(2z(z-x) + (z^2 - x^2 - y^2)).$$

Similarly,

$$V\left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = V(y, 3z + x) + 3V(y, z - x),$$

so that $\Gamma^1_{f,x} = V(y, 3z + x)$ and $\mu^\circ = 3$. It then follows that $\Gamma^1_{f,x}$ is not defined by a single equation inside $\Gamma^2_{f,(x,y)}$

To see that $\widetilde{b}_2(F_{f,\mathbf{0}}) - \widetilde{b}_1(F_{f,\mathbf{0}}) < 0$, we note that, up to analytic isomorphism, f is the homogeneous polynomial $f = (zx - y^2)z$. Consequently, we need only consider the global Milnor fiber of f, i.e., $F_{f,\mathbf{0}}$ is diffeomorphic to $f^{-1}(1)$. Thus, $F_{f,\mathbf{0}}$ is homotopy equivalent to S^1, so that $\widetilde{b}_2(F_{f,\mathbf{0}}) = 0$ and $\widetilde{b}_1(F_{f,\mathbf{0}}) = 1$.

Corollary 4.3.0.5. *The Beta Conjecture is true if the set $\Gamma^2_{f,\mathbf{z}}$ is smooth and transversely intersected by $V(z_0, z_1)$ at the origin. In particular, the Beta Conjecture is true for non-reduced plane curve singularities.*

Proof. Suppose that the cycle $\Gamma^2_{f,\mathbf{z}} = m[V(\mathfrak{p})]$, where $\mathfrak{p}$ is prime. Since the set $\Gamma^2_{f,\mathbf{z}}$ is smooth, $A := \mathcal{O}_{\mathcal{U},\mathbf{0}}/\mathfrak{p}$ is regular and so, in particular, is a UFD. The image of $\partial f/\partial z_1$ in A factors (uniquely), yielding an h as in hypothesis (2) of Theorem 4.3.0.2.

Furthermore, the transversality of $V(z_0, z_1)$ to $\Gamma^2_{f,\mathbf{z}}$ at the origin assures us that, by replacing z_0 by a generic linear combination $az_0 + bz_1$, we obtain hypothesis (1) of Theorem 4.3.0.2. □

Example 4.3.0.6. Consider the case where $f = z^2 + (y^2 - x^3)^2$ on $\mathbb{C}^3$, with coordinate ordering (x, y, z); a quick calculation shows that $\Sigma f = V(z, y^2 - x^3)$. Then,

$$\Gamma^2_{f,(x,y)} = V\left(\frac{\partial f}{\partial z}\right) = V(z)$$

is clearly smooth at the origin and transversely intersected at $\mathbf{0}$ by the line $V(x, y)$, so the hypotheses of Corollary 4.3.0.5 are satisfied. Again, we want to verify by hand that the inequality

$$\lambda^0_{f,x} - \lambda^1_{f,x} \geq \left(\Gamma^2_{f,(x,y)} \cdot V(x,y)\right)_{\mathbf{0}}$$

holds.

First, we have

$$\lambda^0_{f,x} = \left(\Gamma^1_{f,x} \cdot V\left(\frac{\partial f}{\partial x}\right)\right)_{\mathbf{0}} = V(y, z, 2(y^2 - x^3)(-3x^2))_{\mathbf{0}} = V(y, z, x^5)_{\mathbf{0}} = 5,$$

and

$$\lambda^1_{f,x} = \left(\Lambda^1_{f,x} \cdot V(x)\right)_{\mathbf{0}} = V(x, z, y^2 - x^3)_{\mathbf{0}} = V(x, z, y^2)_{\mathbf{0}} = 2.$$

On the other hand, we have $\left(\Gamma^2_{f,(x,y)} \cdot V(x,y)\right)_{\mathbf{0}} = V(x, y, z)_{\mathbf{0}} = 1$, and we see again that the desired inequality holds.

In the case where f defines non-reduced plane curve singularity, there is a nice explicit formula for β_f, which we will derive in Subsection 4.3.1.

4.3.1 Non-reduced Plane Curves

By Corollary 4.3.0.5, the Beta Conjecture is true for non-reduced plane curve singularities. However, in that special case, we may calculate β_f explicitly.

Let $\mathcal{U}$ be an open neighborhood of the origin in $\mathbb{C}^2$, with coordinates (x, y).

Proposition 4.3.1.1. *Suppose that f is of the form $f = g(x,y)^p h(x,y)$, where $g : (\mathcal{U}, \mathbf{0}) \to (\mathbb{C}, \mathbf{0})$ is irreducible, g does not divide h, and $p > 1$. Then,*

$$\beta_f = \begin{cases} (p+1)V(g,h)_{\mathbf{0}} + p\mu_{\mathbf{0}}(g) + \mu_{\mathbf{0}}(h) - 1, & \textit{if } h(\mathbf{0}) = 0\textit{; and} \\ p\mu_{\mathbf{0}}(g), & \textit{if } h(\mathbf{0}) \neq 0. \end{cases}$$

Thus, $\beta_f = 0$ implies that Σf is smooth at $\mathbf{0}$.

Proof. After a possible linear change of coordinates, we may assume that the first coordinate x satisifes $\dim_{\mathbf{0}} \Sigma(f_{|_{V(x)}}) = 0$, so that $\dim_{\mathbf{0}} V(g,x) = \dim_{\mathbf{0}} V(h,x) = 0$ as well.

As germs of sets at $\mathbf{0}$, the critical locus of f is simply $V(g)$. As cycles,

$$\begin{aligned} V\left(\frac{\partial f}{\partial y}\right) &= \Gamma^1_{f,x} + \Lambda^1_{f,x} = V\left(phg^{p-1}\frac{\partial g}{\partial y} + g^p\frac{\partial h}{\partial y}\right) \\ &= V\left(ph\frac{\partial g}{\partial y} + g\frac{\partial h}{\partial y}\right) + (p-1)V(g), \end{aligned}$$

so that $\Gamma^1_{f,x} = V\left(ph\frac{\partial g}{\partial y} + g\frac{\partial h}{\partial y}\right)$ and Σf consists of a single component $C = V(g)$. It is a quick exercise to show that, for g irreducible, g does not divide $\frac{\partial g}{\partial y}$, and so the nearby Milnor number is precisely $\mu^\circ_C = (p-1)$ along $V(g)$.

Suppose first that $h(\mathbf{0}) = 0$.

Then, by Proposition 4.1.0.8,

$$\lambda^0_{f,x} - \lambda^1_{f,x} = \left(\Gamma^1_{f,x} \cdot V(f)\right)_{\mathbf{0}} - \mu_{\mathbf{0}}\left(f_{|_{V(x)}}\right).$$

We then expand the terms on the right hand side, as follows:

$$\begin{aligned}\left(\Gamma^1_{f,x}\cdot V(f)\right)_{\mathbf{0}} &= p\left(\Gamma^1_{f,x}\cdot V(g)\right)_{\mathbf{0}} + \left(\Gamma^1_{f,x}\cdot V(h)\right)_{\mathbf{0}}\\ &= pV\left(g, h\frac{\partial g}{\partial y}\right)_{\mathbf{0}} + V\left(h, g\frac{\partial h}{\partial y}\right)_{\mathbf{0}}\\ &= (p+1)V(g,h)_{\mathbf{0}} + pV\left(g, \frac{\partial g}{\partial y}\right)_{\mathbf{0}} + V\left(h, \frac{\partial h}{\partial y}\right)_{\mathbf{0}}.\end{aligned}$$

Since $\dim_{\mathbf{0}} V(g,x) = 0$ and $\dim_{\mathbf{0}} V(h,x) = 0$, the relative polar curves of g and h with respect to x are, respectively, $\Gamma^1_{g,x} = V\left(\frac{\partial g}{\partial y}\right)$ and $\Gamma^1_{h,x} = V\left(\frac{\partial h}{\partial y}\right)$. We can therefore apply Teissier's trick to this last equality to obtain

$$\begin{aligned}\left(\Gamma^1_{f,x}\cdot V(f)\right)_{\mathbf{0}} &= (p+1)V(g,h)_{\mathbf{0}} + p\left[V\left(\frac{\partial g}{\partial y}, x\right)_{\mathbf{0}} + \mu_{\mathbf{0}}(g)\right] + \left[V\left(\frac{\partial h}{\partial y}, x\right)_{\mathbf{0}} + \mu_{\mathbf{0}}(h)\right]\\ &= (p+1)V(g,h)_{\mathbf{0}} + p\mu_{\mathbf{0}}(g) + pV(g,x)_{\mathbf{0}} + V(h,x)_{\mathbf{0}} - (p+1).\end{aligned}$$

Next, we calculate the Milnor number of the restriction of f to $V(x)$:

$$\mu_{\mathbf{0}}\left(f_{|_{V(x)}}\right) = V\left(\frac{\partial f}{\partial y}, x\right)_{\mathbf{0}} = \left(\Gamma^1_{f,x}\cdot V(x)\right)_{\mathbf{0}} + (p-1)V(g,x)_{\mathbf{0}}.$$

Substituting these equations back into our initial identity, we obtain the following:

$$\begin{aligned}\lambda^0_{f,x} - \lambda^1_{f,x} &= (p+1)V(g,h)_{\mathbf{0}} + V(g,x)_{\mathbf{0}} + V(h,x)_{\mathbf{0}}\\ &\quad + p\mu_{\mathbf{0}}(g) + \mu_{\mathbf{0}}(h) - \left(\Gamma^1_{f,x}\cdot V(x)\right)_{\mathbf{0}} - (p+1).\end{aligned}$$

We now wish to show that $\left(\Gamma^1_{f,x}\cdot V(x)\right)_{\mathbf{0}} = V(gh,x)_{\mathbf{0}} - 1$. To see this, we first recall that

$$\left(\Gamma^1_{f,x}\cdot V(x)\right)_{\mathbf{0}} = \mathrm{mult}_y\left\{\left(ph\cdot\frac{\partial g}{\partial y}\right)_{|_{V(x)}} + \left(g\cdot\frac{\partial h}{\partial y}\right)_{|_{V(x)}}\right\},$$

where $g_{|_{V(x)}}$ and $h_{|_{V(x)}}$ are (convergent) power series in y with constant coefficients. If the lowest-degree terms in y of $\left(ph\frac{\partial g}{\partial y}\right)_{|_{V(x)}}$ and $\left(g\frac{\partial h}{\partial y}\right)_{|_{V(x)}}$ do not cancel each other out, then the y-multiplicity of their sum is the minimum of their respective y-multiplicities, both of which equal $V(gh,x)_{\mathbf{0}} - 1$. We must show that no such cancellation can occur. To this end, let $g_{|_{V(x)}} = \sum_{i\geq n} a_i y^i$ and $h_{|_{V(x)}} = \sum_{i\geq m} b_i y^i$ be power series representations in y, where

$n = \operatorname{mult}_y g_{|V(x)}$ and $m = \operatorname{mult}_y h_{|V(x)}$ (so that $a_n, b_m \neq 0$). Then, a quick computation shows that the lowest-degree term of $\left(ph\frac{\partial g}{\partial y}\right)_{|V(x)}$ is $pn\, a_n b_m$, and the lowest-degree term of $\left(g\frac{\partial h}{\partial y}\right)_{|V(x)}$ is $m\, a_n b_m$. Consequently, no cancellation occurs, and thus $\left(\Gamma^1_{f,x} \cdot V(x)\right)_{\mathbf{0}} = V(gh, x)_{\mathbf{0}} - 1 = n + m - 1$.

Therefore, we conclude that

$$\beta_f = (p+1)V(g,h)_{\mathbf{0}} + p\mu_{\mathbf{0}}(g) + \mu_{\mathbf{0}}(h) - 1.$$

Since $V(g)$ and $V(h)$ have a non-empty intersection at $\mathbf{0}$, the intersection number $V(g,h)_{\mathbf{0}}$ is greater than one (so that $\beta_f > 0$).

Suppose now that $h(\mathbf{0}) \neq 0$. Then, from the above calculations, we find

$$\begin{aligned} \left(\Gamma^1_{f,x} \cdot V(f)\right)_{\mathbf{0}} &= p\mu_{\mathbf{0}}(g) + pV(g,x)_{\mathbf{0}} - (p+1), \text{ and} \\ \mu_{\mathbf{0}}\left(f_{|V(x)}\right) &= pV(g,x)_{\mathbf{0}} - 1 \end{aligned}$$

so that $\beta_f = p\mu_{\mathbf{0}}(g)$.

Recall that, as $\Sigma f = V(g)$, the critical locus of f is smooth at $\mathbf{0}$ if and only if $V(g)$ is smooth at $\mathbf{0}$; equivalently, if and only if the Milnor number of g at $\mathbf{0}$ vanishes. Hence, when Σf is *not* smooth at $\mathbf{0}$, $\mu_{\mathbf{0}}(g) > 0$, and we find that $\beta_f > 0$, as desired. $\square$

Remark 4.3.1.2. Suppose that $f(x,y)$ is of the form $f = gh$, where g and h are relatively prime, and both have isolated critical points at the origin. Then, f has an isolated critical point at $\mathbf{0}$ as well, and the same computation in Proposition 4.3.1.1 (for $\mu_{\mathbf{0}}(f)$ instead of β_f) yields the formula

$$\mu_{\mathbf{0}}(f) = 2V(g,h)_{\mathbf{0}} + \mu_{\mathbf{0}}(g) + \mu_{\mathbf{0}}(h) - 1.$$

Thus, the formula for β_f in the non-reduced case collapses to the "expected value" of $\mu_{\mathbf{0}}(f)$ exactly when $p = 1$ and f has an isolated critical point at the origin.

Appendix A

The Lê Cycles and Relative Polar Varieties

The Lê numbers of a function with a non-isolated critical locus are the fundamental invariants we consider in this paper. First defined by Massey in [48] and [49], these numbers generalize the Milnor number of a function with an isolated critical point.

The Lê cycles and numbers of g are classically defined with respect to a **prepolar-tuple** of linear forms $\mathbf{z} = (z_0, \cdots, z_n)$; loosely, these are linear forms that transversely intersect all strata of a good stratification of $V(g)$ near $\mathbf{0}$ (see, for example, Definition 1.26 of [42]). The purpose of Proposition 2.1.0.10 in Section 2.1 is to replace the assumption of prepolar-tuples with IPA tuples.

Definition A.0.0.1. *The* k***-dimensional relative polar variety of*** g ***with respect to*** $\mathbf{z}$*, at the origin, denoted* $\Gamma^k_{g,\mathbf{z}}$*, consists of those components of the analytic cycle* $V\left(\frac{\partial g}{\partial z_k}, \cdots, \frac{\partial g}{\partial z_n}\right)$ *at the origin which* are not *contained in* Σg.

Definition A.0.0.2. *The* k***-dimensional Lê cycle of*** g ***with respect to*** $\mathbf{z}$*, at the origin, denoted* $\Lambda^k_{g,\mathbf{z}}$*, consists of those components of the analytic cycle* $\Gamma^{k+1}_{g,\mathbf{z}} \cdot V\left(\frac{\partial g}{\partial z_k}\right)$ *which* are *contained in* Σg.

Definition A.0.0.3. *The k-**dimensional Lê number of** g **at** $p = (p_0, \cdots, p_n)$ **with respect to** $\mathbf{z}$, denoted $\lambda^k_{g,\mathbf{z}}(p)$, is equal to the intersection number*

$$\left(\Lambda^k_{g,\mathbf{z}} \cdot V(z_0 - p_0, \cdots, z_{k-1} - p_{k-1})\right)_p ,$$

provided this intersection is purely zero-dimensional at p.

Example A.0.0.4. When g has an isolated critical point at the origin, the only non-zero Lê number of g is $\lambda^0_{g,\mathbf{z}}(\mathbf{0})$. In this case, we have:

$$\begin{aligned} \lambda^0_{g,\mathbf{z}}(\mathbf{0}) &= \left(\Lambda^0_{g,\mathbf{z}} \cdot \mathcal{U}\right)_{\mathbf{0}} \\ &= V\left(\frac{\partial g}{\partial z_0}, \cdots, \frac{\partial g}{\partial z_n}\right)_{\mathbf{0}}, \end{aligned}$$

i.e., the 0-dimensonal Lê number of g is just the multiplicity of the Jacobian scheme. In the case of an isolated critical point, this is the Milnor number of g at $\mathbf{0}$.

Example A.0.0.5. Suppose now that $\dim_{\mathbf{0}} \Sigma g = 1$. Then, the only non-zero Lê numbers of g are $\lambda^0_{g,\mathbf{z}}(\mathbf{0})$ and $\lambda^1_{g,\mathbf{z}}(p)$ for $p \in \Sigma g$.

At $\mathbf{0}$, we have

$$\begin{aligned} \lambda^1_{g,\mathbf{z}}(\mathbf{0}) &= \left(\Lambda^1_{g,\mathbf{z}} \cdot V(z_0)\right)_{\mathbf{0}} \\ &= \left(V\left(\frac{\partial g}{\partial z_1}, \cdots, \frac{\partial g}{\partial z_n}\right) \cdot V(z_0)\right)_{\mathbf{0}} \\ &= \sum_{q \in B_\epsilon \cap V(z_0 - q_0) \cap \Sigma g} \left(V\left(\frac{\partial g}{\partial z_1}, \cdots, \frac{\partial g}{\partial z_n}\right) \cdot V(z_0 - q_0)\right)_q \\ &= \sum_{q \in B_\epsilon \cap V(z_0 - q_0) \cap \Sigma g} \mu_q\left(g_{|V(z_0 - q_0)}\right) \end{aligned}$$

where the second to last line is the dynamic intersection property for proper intersections.

After rearranging the terms in the last line, we find

$$\lambda^1_{g,\mathbf{z}}(\mathbf{0}) = \sum_{C \subseteq \Sigma g \text{ irred. comp.}} \overset{\circ}{\mu}_C \left(C \cdot V(z_0)\right)_{\mathbf{0}} ,$$

where the sum is indexed over the collection of irreducible components of Σg at the origin, and $\overset{\circ}{\mu}_C$ denotes the generic transversal Milnor number of g along C.

Appendix B

Singularities of Maps

Our primary references for this appendix are [54], [11], and [52].

Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, \mathbf{0})$ be a holomorphic map (multi-)germ, with S a finite subset of $\mathbb{C}^n$. Then, the group of biholomorphisms $\mathrm{Diff}(N, S)$ from $\mathbb{C}^n$ to $\mathbb{C}^n$ (preserving S), acts on f on the left by pre-composition; similarly, the group of biholomorphisms $\mathrm{Diff}(\mathbb{C}^p, \mathbf{0})$ from $\mathbb{C}^p$ to to $\mathbb{C}^p$ (preserving the origin), acts on f on the right by composition. Thus, we have a group action of $\mathcal{A} := \mathrm{Diff}(\mathbb{C}^n, S) \times \mathrm{Diff}(\mathbb{C}^p, \mathbf{0})$ on the space of all holomorphic maps $\mathcal{O}(n, p)$ from $(\mathbb{C}^n, S)$ to $(\mathbb{C}^p, \mathbf{0})$:

$$\begin{aligned} \mathcal{A} \times \mathcal{O}(n,p) &\to \mathcal{O}(n,p) \\ (\Phi, \Psi) * f &= \Phi \circ f \circ \Psi^{-1}. \end{aligned}$$

Clearly, this group action defines an equivalence relation on $\mathcal{O}(n, p)$, where $f \sim g$ if there exists $(\Phi, \Psi) \in \mathcal{A}$ for which $\Phi^{-1} \circ f \circ \Psi = g$. Let $\mathcal{A}_e$ denote the pseudo-group gotten by allowing non-origin preserving equivalences, and $\mathcal{O}_e(n, p)$ the space of map-germs at the origin, but not necessarily origin-preserving.

Definition B.0.0.1. *A d-**parameter unfolding** of f is a map germ*

$$F : (\mathbb{C}^d \times \mathbb{C}^n, \{\mathbf{0}\} \times S) \to (\mathbb{C}^d \times \mathbb{C}^p, \mathbf{0})$$

of the form

$$F(\mathbf{t}, \mathbf{z}) = (\mathbf{t}, \widetilde{f}(\mathbf{t}, \mathbf{z})),$$

such that $\widetilde{f}(\mathbf{0}, \mathbf{z}) = f(\mathbf{z})$*, and* $\mathbf{t} = (t_1, \cdots, t_d)$ *are coordinates on* $\mathbb{C}^d$*. We also write* $f_{\mathbf{t}}(\mathbf{z}) := \widetilde{f}(\mathbf{t}, \mathbf{z})$*, so that* $f_0 = f$*.*

We say F *is a* ***trivial unfolding*** *of* f *if there are* d*-parameter unfoldings of the identity on* $\mathbb{C}^n$ *and* $\mathbb{C}^p$*, say* Φ *and* Ψ*, respectively, such that* $\Phi \circ F \circ \Psi^{-1} = (id, f)$*.*

Definition B.0.0.2. *We say* $f \in \mathcal{O}_e(n, p)$ *is* ***stable*** *if every unfolding of* f *is trivial.*

Definition B.0.0.3. *We say an unfolding* $F : (\mathbb{C}^d \times \mathbb{C}^n, \{0\} \times S) \to (\mathbb{C}^d \times \mathbb{C}^p, \mathbf{0})$*,* $F(\mathbf{t}, \mathbf{z}) = (\mathbf{t}, f_{\mathbf{t}}(\mathbf{z}))$ *of* f *is a* ***stable unfolding*** *(or, a* ***stabilization****) of* f *if* $f_{\mathbf{t}}$ *is stable for all* $t \neq 0$*.*

Definition B.0.0.4. *We say that a map* $f \in \mathcal{O}(n, p)$ *is* ***finitely determined*** *if there exists an integer* k *such that any* $g \in \mathcal{O}(n, p)$ *which has the same* k*-jet as* f *satisfies* $f \sim g$*. That is, if, for all* $x \in S$*, the derivatives of* f *and* g *at* x *of order* $\leq k$ *are the same (with respect to a system of coordinates at* x *and* y*).*

We primarily care about (one-parameter) stabilizations of finitely-determined map germs for the fact that these maps all have isolated instabilities at the origin (see Section 2.2). In general, we have the following remark.

Remark B.0.0.5. Suppose, that F is a stable one-parameter unfolding of a finite map f, and that $h : (\operatorname{im} F, \mathbf{0}) \to (\mathbb{C}, 0)$ is the projection onto the unfolding parameter. Then a point $x \in V(h)$ is a point in the image of f. If f is stable at x, then h is locally a topologically trivial fibration in a neighborhood of x; consequently, the Milnor fiber is contractible, and $x \notin \Sigma_{\text{top}} h$. Thus, $\Sigma_{\text{top}} h$ is contained in the unstable locus of F_0. We will need this observation in Section 2.2.

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