Hypersurface Normalizations and Numerical Invariants

Thesis Defense

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Outline

- Motivation and a Result of Milnor
- Deforming Parameterized Hypersurfaces
- Rational Homology Manifolds and $\mathbf{N}_X^\bullet$
- Mixed Hodge Modules and $\mathbf{N}_X^\bullet$
- Future Directions

Dramatis Personæ

- The central object of study for this work is a perverse sheaf called the **comparison complex**, first defined and explored by H. and David Massey in [7] (where we originally referred to it as the **multiple-point complex**), and in several subsequent papers by H. [4],[5],[6] and Massey [9].

- This perverse sheaf, denoted $\mathbf{N}^\bullet_X$, is defined on any pure-dimensional (locally reduced) complex analytic space X for which the constant sheaf $\mathbb{Q}^\bullet_X[\dim X]$ is perverse.

- $\mathbf{N}^\bullet_X$ "compares" the two fundamental perverse sheaves on such a space, the **constant sheaf** $\mathbb{Q}^\bullet_X[n]$ and the **intersection cohomology complex** $\mathbf{I}^\bullet_X$ with constant coefficients, in a way that detects (some of) the singularities of X.

Starting Point: Milnor's Result

- Suppose $(V(f_0), \mathbf{0}) \subseteq (\mathbb{C}^2, \mathbf{0})$ is plane curve singularitiy in $\mathbb{C}^2$, with r irreducible components at the origin.
- Then, by a well-known result of John Milnor [12], the Milnor number $\mu_{\mathbf{0}}(f_0)$ is related to the number of double points δ which occur in a generic (stable) deformation of f_0 by

$$\mu_{\mathbf{0}}(f_0) = 2\delta - r + 1.$$

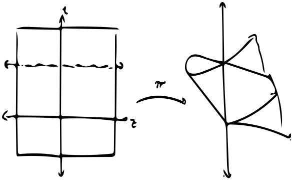
- We wish to generalize this formula to deformations of hypersurfaces with codimension-one singularities.

Theorem (Milnor '68 [12])

Suppose that π is a one-parameter unfolding of a parameterization π_0 of a plane curve singularity $V(f_0)$ with r irreducible components at the origin, and set $\operatorname{im}\pi = V(f)$. Let t be the unfolding parameter and suppose that the only singularities of the complex link $\mathbb{L}_{V(f),\mathbf{0}}$ are nodes, and that there are δ of them. Let $f_0 := f_{|_{V(t)}}$.

Then, the Milnor number of f_0 is given by the formula:

$$\mu_{\mathbf{0}}(f_0) = 2\delta - r + 1.$$

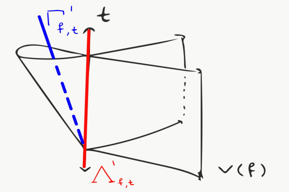


Quick Proof of Milnor's Result

- The deformation of the function f_0 is an analytic function f defining the surface $\operatorname{im}\pi = V(f)$ in $\mathbb{C}^3$, together with a choice of "nice" linear form t on $\mathbb{C}^3$ provided by the unfolding parameter of the parameterization. As in the theorem, we have $f_0 := f_{|_{V(t)}}$.
- Recall the following formula for the Milnor number of f_0 at $\mathbf{0}$:

$$\mu_{\mathbf{0}}(f_0) = \left(\Gamma^1_{f,t} \cdot V(t)\right)_{\mathbf{0}} + \left(\Lambda^1_{f,t} \cdot V(t)\right)_{\mathbf{0}},$$

where $\Gamma^1_{f,t}$ **is the relative polar curve of** g **with respect to** t, and $\Lambda^1_{f,t}$ **is the one-dimensional Lê cycle of** f **with respect to** t.



Quick Proof of Milnor's Result

- Note then that, for t_0 small and non-zero, we can identify the deformed curve $V(f_{t_0})$ with the complex link $\mathbb{L}_{V(f),\mathbf{0}}$ of $V(f)$ at the origin.

- By assumption, the only singularities of $\mathbb{L}_{V(f),\mathbf{0}}$ are nodes, **each of which necessarily has Milnor number equal to 1**, so we immediately have $\left(\Lambda^1_{f,t} \cdot V(t)\right)_{\mathbf{0}} = \delta$.
- Since the unfolding π has an isolated instability at $\mathbf{0}$, $\mu_{\mathbf{0}}(t_{|_{V(f)}})$ is defined and equal **the number of 1-spheres in the homotopy-type of the complex link**, given by $\left(\Gamma^1_{f,t} \cdot V(t)\right)_{\mathbf{0}}$.

Quick Proof of Milnor's Result

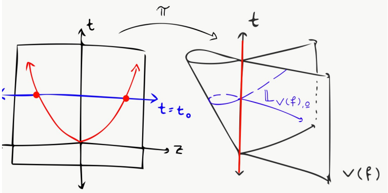
- Now, by [7],

$$\mu_{\mathbf{0}}(t_{|_{V(f)}}) = -r + 1 + \sum_{k\geq 2}(k-1)\chi(V(f)_k \cap \mathbb{L}_{V(f),\mathbf{0}}),$$

where $V(f)_k := \{p \in V(f) \,|\, |\pi^{-1}(p)| = k\}$.

- By assumption, $\chi(V(f)_2 \cap \mathbb{L}_{V(f),\mathbf{0}})$ is the only non-zero summand in the above equation, and it is immediately seen to be the number of double points of $V(f_0)$ appearing in a stable perturbation. Thus,

$$\mu_{\mathbf{0}}\left(f_{|_{V(t)}}\right) = 2\delta - r + 1.$$



Rewriting the Formula

- **What if we didn't have such a "stable deformation" of $V(f_0)$?** That is, what if we didn't know that the origin $\mathbf{0} \in V(f_0)$ splits into δ nodes? We can still use the techniques of [7] in this situation. In this case, we have

$$\mu_{\mathbf{0}}(f_0) = -r + 1 + \sum_{p \in B_\epsilon \cap V(t - t_0)} (\mu_p(f_{t_0}) + m(p))$$

  where $m(p) := |\pi^{-1}(p)| - 1$.
- **The main idea** behind generalizing Milnor's formula to higher dimensions is that, when the hypersurface $V(f)$ is parameterized, there is a natural perverse sheaf on $V(f)$ that generalizes the function $m(p)$.

Parameterizations

- Let $\pi : (\mathbb{D} \times \mathcal{W}, \{0\} \times S) \to (V(f), \mathbf{0})$ be a surjective finite map which is **generically one-to-one**, and is further of the form

$$\pi(t, \mathbf{z}) = (t, \pi_t(\mathbf{z})),$$

  where π_0 is a generically one-to-one parameterization of $V(f_0)$. Here $\mathcal{W}$ is an open subset of $\mathbb{C}^{n-1}$, $\pi^{-1}(\mathbf{0}) = S$, and $\mathbb{D}$ is an open disk around the origin in $\mathbb{C}$.
  This means that π **is a one-parameter unfolding of** π_0.
- Then, in the Abelian category of perverse sheaves on $V(f)$, there is a canonical surjective morphism $\mathbb{Z}^{\bullet}_{V(f)}[n] \xrightarrow{\Delta} \pi_* \mathbb{Z}^{\bullet}_{\mathbb{D}\times\mathcal{W}}[n]$. We let $\mathbf{N}^{\bullet}_{V(f)}$ be the kernel of this morphism, so that we have a **short exact sequence of perverse sheaves**

$$0 \to \mathbf{N}^{\bullet}_{V(f)} \to \mathbb{Z}^{\bullet}_{V(f)}[n] \xrightarrow{\Delta} \pi_* \mathbb{Z}^{\bullet}_{\mathbb{D}\times\mathcal{W}}[n] \to 0.$$

The Multiple Point Complex

The complex $\mathbf{N}^\bullet_{V(f)}$ was called the **multiple-point complex** of $V(f)$, and is supported on the image multiple-point set

$$D := \overline{\{x \in V(f) \,|\, |\pi^{-1}(x)| > 1\}}.$$

The multiple-point complex has several useful properties:

- It is a perverse sheaf on $V(f)$.
- It has nonzero stalk cohomology only in degree $-(n-1)$, where $n = \dim_{\mathbf{0}} V(f)$.
- In degree $-(n-1)$, the stalk cohomology is very easy to describe: for $p \in V(f)$,

  $$H^{-(n-1)}(\mathbf{N}^\bullet_{V(f)})_p \cong \mathbb{Z}^{m(p)}.$$

  where $m(p) := |\pi^{-1}(p)| - 1$, as before.

What would a generalization look like?

- One of the trade-offs for parameterizing $V(f)$ is that **one is forced to have codimension-one singularities**; that is, $\operatorname{supp} \mathbf{N}^{\bullet}_{V(f)} = D \subseteq \Sigma f$, and D is purely $(n-1)$-dimensional.

- One natural generalization of the Milnor number to higher-dimensional singularities are the Lê numbers–so we will express the **Lê numbers** of the $t = 0$ slice in terms of the Lê numbers of the $t \neq 0$ slice, together with the **characteristic polar multiplicities of $\mathbf{N}^{\bullet}_{V(f)}$** (discussed below).

- **What sort of deformation do we want?** We don't necessarily have a deformation into something as nice as double-points. We choose the notion of an **IPA-deformation**–these are deformations which, intuitively, are those where the only "interesting" behavior happens at the origin.

Characteristic Polar Multiplicities and Lê Numbers

- For any perverse sheaf $\mathbf{P}^\bullet$ on an analytic subset of $\mathbb{C}^N$, **the characteristic polar multiplicities of $\mathbf{P}^\bullet$** with respect to a "nice" choice of linear forms $\mathbf{z} = (z_0, \cdots, z_s)$, denoted $\lambda^i_{\mathbf{P}^\bullet, \mathbf{z}}(p)$ (defined by Massey in [11]) are non-negative integer-valued functions that mimic the construction of the Lê numbers associated to non-isolated hypersurface singularities.

- Indeed, one has the equalities

$$\lambda^i_{f,\mathbf{z}}(p) = \lambda^i_{\phi_f[-1]\mathbb{Z}^\bullet_{\mathbb{C}^N}[N]}(p),$$

for $0 \leq i \leq \dim_{\mathbf{0}} \Sigma f$ and all p in some open neighborhood of $\mathbf{0}$ in $\mathbb{C}^N$.

A Conserved Quantity

Theorem (H., '17 [4])

Suppose that $\pi : (\mathbb{D} \times \mathcal{W}, \{0\} \times S) \to (V(f), \mathbf{0})$ *is a one-parameter unfolding with an isolated instability of a parameterized hypersurface* $\operatorname{im}\pi_0 = V(f_0)$*. Suppose further that* $\mathbf{z} = (z_1, \cdots, z_n)$ *is chosen such that* $\mathbf{z}$ *is an IPA-tuple for* $f_0 = f_{|_{V(t)}}$ *at* $\mathbf{0}$*.*

Then, the following relationship holds for the Lê numbers of f_0 *with respect to* $\mathbf{z}$ *at* $\mathbf{0}$*: for* $0 < |t_0| \ll \epsilon \ll 1$*,*

$$\lambda^i_{f_0,\mathbf{z}}(\mathbf{0}) + \lambda^i_{\mathbf{N}^\bullet_{V(f_0)},\mathbf{z}}(\mathbf{0}) = \sum_{p \in B_\epsilon \cap V(t-t_0,z_1,z_2,\cdots,z_i)} \left(\lambda^i_{f_{t_0},\mathbf{z}}(p) + \lambda^i_{\mathbf{N}^\bullet_{V(f_{t_0})},\mathbf{z}}(p) \right)$$

Sketch of Theorem

- A large part of the proof is verifying that the IPA-condition is sufficient to simultaneously guarantee the numbers $\lambda^i_{f,(t,\mathbf{z})}(\mathbf{0})$ and $\lambda^i_{\mathbf{N}^\bullet_{V(f)},(t,\mathbf{z})}(\mathbf{0})$ are defined.
- After applying the dynamic intersection property to rewrite $\lambda^i_{f_0,\mathbf{z}}(\mathbf{0})$ as a sum in the $t \neq 0$ slice, it suffices to prove

$$\lambda^0_{\mathbf{N}^\bullet_{V(f)},(t,\mathbf{z})} = -\lambda^0_{\mathbf{N}^\bullet_{V(f_0)},\mathbf{z}}(\mathbf{0}) + \sum_{p \in B_\epsilon \cap V(t-t_0)} \lambda^0_{\mathbf{N}^\bullet_{V(f_{t_0})},\mathbf{z}}(p).$$

- Since $(t, \mathbf{z})$ is an IPA-tuple for f at $\mathbf{0}$, we have

$$\lambda^0_{\mathbf{N}^\bullet_{V(f_0)},\mathbf{z}}(\mathbf{0}) = \lambda^1_{\mathbf{N}^\bullet_{V(f)},(t,\mathbf{z})}(\mathbf{0}) - \lambda^0_{\mathbf{N}^\bullet_{V(f)},(t,\mathbf{z})}(\mathbf{0}).$$

  The claim follows from again applying the dynamic intersection property for proper intersections with $\Lambda^1_{\mathbf{N}^\bullet_{V(f)},(t,\mathbf{z})}$.

Theorem (H.,'19 [4])

Suppose $\pi : (\mathbb{D} \times \mathbb{C}^2, \{\mathbf{0}\} \times S) \to (\mathbb{C}^4, \mathbf{0})$ *is a one-parameter unfolding of a finitely-determined map germ* $\pi_0 : (\mathbb{C}^2, S) \to (\mathbb{C}^3, \mathbf{0})$ *parameterizing a surface* $V(f_0) \subseteq \mathbb{C}^3$*. Then,*

$$\lambda^0_{\mathbf{N}^\bullet_{V(f_0)}, \mathbf{z}}(\mathbf{0}) = T + C - \delta + P$$

where T*,* C*,* δ*, and* P *denote the number of triple points, cross caps,* A_1*-singularities appearing in a stable deformation of* $V(f_0)$*, respectively, and if* $V(f) = \operatorname{im} \pi$*,* P *denotes the number of intersection points of the absolute polar curve* $\Gamma^1_{(t,z)}(\Sigma f)$ *with a hyperplane* $V(z)$ *on* $\mathbb{C}^4$ *for which* (t, z) *is an IPA-tuple for* f *at* $\mathbf{0}$*.*

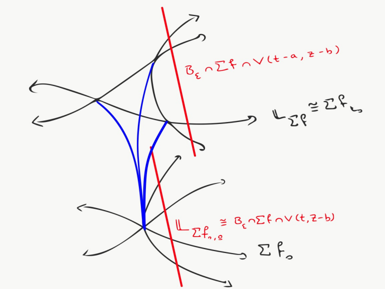
Corollary

Let π*,* T, C*, and* δ *be as above. Then,*

$$-\chi(\mathbf{N}^\bullet_{V(f_0)})_\mathbf{0} = |\pi_0^{-1}(\mathbf{0})| - 1 = \chi(\mathbb{L}_{\Sigma f, \mathbf{0}}) + T - C + \delta,$$

Sketch of Corollary

- We compute the Euler characteristic of the pair $\chi(\mathbb{L}_{\Sigma f,\mathbf{0}}, \mathbb{L}_{\Sigma f_0,\mathbf{0}})$.
- This pair of subspaces makes sense, using the fact that f is an IPA-deformation of f_0, and the complex link $\mathbb{L}_{\Sigma f_0,\mathbf{0}}$ of Σf_0 is a finite set of points, and their multiplicity is unchanged as one moves in the t direction away from the origin, pictured below:

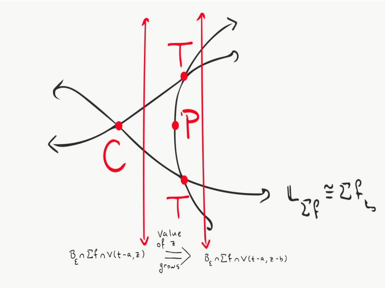


Sketch of Corollary

- Thus, we can identify $\mathbb{L}_{\Sigma f_0,\mathbf{0}} = B_\epsilon \cap \Sigma f \cap V(t, z-b)$ with $B_\epsilon \cap \Sigma f \cap V(t-a, z-b)$ for $0 < |a| \ll |b| \ll \epsilon \ll 1$. Consequently,

$$\chi(\mathbb{L}_{\Sigma f,\mathbf{0}}, \mathbb{L}_{\Sigma f_0,\mathbf{0}}) = \chi(\phi_z[-1]\mathbb{Z}^\bullet_{\mathbb{L}_{\Sigma f,\mathbf{0}}}[1])_{\mathbf{0}} = \sum_p \lambda^0_{\mathbb{Z}^\bullet_{\Sigma f_{t_0}}[1],z}(p).$$

- As the value of z grows from 0 to b, we pick up cohomological contributions (in the form of a non-zero multiplicity $\lambda^0_{\mathbb{Z}^\bullet_{\Sigma f_{t_0}}[1],z}(p)$) as we pass through points of the curves of triple points, cross caps, and the absolute polar curve with respect to (t, z), pictured below:



Sketch of Corollary

- At triple points, $\lambda^0_{\mathbb{Z}^\bullet_{\Sigma f_{t_0}}[1],z}(p) = 2$, and at cross caps $\lambda^0_{\mathbb{Z}^\bullet_{\Sigma f_{t_0}}[1],z}(p) = 0$. We count the contribution from the absolute polar curve as $P = \left(\Gamma^1_{(t,z)}(\Sigma f) \cdot V(z)\right)_{\mathbf{0}}$.
- Thus,

$$\begin{aligned} 2T + P = \chi(\mathbb{L}_{\Sigma f,\mathbf{0}}, \mathbb{L}_{\Sigma f_0,\mathbf{0}}) &= \chi(\mathbb{L}_{\Sigma f,\mathbf{0}}) - \chi(\mathbb{L}_{\Sigma f_0,\mathbf{0}}) \\ &= \chi(\mathbb{L}_{\Sigma f,\mathbf{0}}) - \lambda^1_{\mathbf{N}^\bullet_{V(f_0)},z}(\mathbf{0}). \end{aligned}$$

- Solving for P and plugging the resulting expression into the Theorem gives the result.

Rational Homology Manifolds

How general are the techniques used to prove these theorems?

- The **fundamental short exact sequence**

$$0 \to \mathbf{N}^\bullet_X \to \mathbb{Q}^\bullet_X[n] \to \mathbf{I}^\bullet_X \to 0$$

  makes sense whenever $\mathbb{Q}^\bullet_X[n]$ is perverse, where $\mathbf{I}^\bullet_X$ is intersection cohomology with $\mathbb{Q}$ constant coefficients.
- If $\pi : \widetilde{X} \to X$ is a **small map** (in particular, if $\widetilde{X}$ is the normalization of X), then we have $\pi_*\mathbf{I}^\bullet_{\widetilde{X}} \cong \mathbf{I}^\bullet_X$, and this short exact sequence becomes

$$0 \to \mathbf{N}^\bullet_X \to \mathbb{Q}^\bullet_X[n] \to \pi_*\mathbf{I}^\bullet_{\widetilde{X}} \to 0,$$

  which we refer to as the **fundamental short exact sequence of the normalization.**

Rational Homology Manifolds

- Looking at the fundamental short exact sequence, one notices immediately that $\mathbb{Q}_X^\bullet[n] \cong \mathbf{I}_X^\bullet$ if and only if $\mathbf{N}_X^\bullet = 0$; that is, the space X is a rational homology manifold (or, a $\mathbb{Q}$-**homology manifold**) precisely when the complex $\mathbf{N}_X^\bullet$ vanishes.
- It is then natural to ask that, given the normalization $\pi : \widetilde{X} \to X$ and the resulting fundamental short exact sequence

$$0 \to \mathbf{N}_X^\bullet \to \mathbb{Q}_X^\bullet[n] \to \pi_* \mathbf{I}_{\widetilde{X}}^\bullet \to 0,$$

  is there a similar result relating $\mathbf{N}_X^\bullet$ to whether or not $\widetilde{X}$ is a $\mathbb{Q}$-homology manifold?

Theorem (H., 2018 [5])

$\widetilde{X}$ is a $\mathbb{Q}$-homology manifold if and only if $\mathbf{N}_X^\bullet$ has stalk cohomology concentrated in degree $-n+1$; i.e., for all $p \in X$, $H^k(\mathbf{N}_X^\bullet)_p$ is non-zero only possibly when $k = -n+1$.

A Trivial, Non-Trivial Example

- Let $f(x,y,z) = xz^2 - y^2(y+x^3)$, so that $X = V(f) \subseteq \mathbb{C}^3$ has $\Sigma f = V(y,z)$. Then, if

$$\widetilde{X} = V(u^2 - x(y+x^3), uy - xz, uz - y(y+x^3)) \subseteq \mathbb{C}^4,$$

the projection map $\pi : \widetilde{X} \to X$ is the normalization of X.

- It is easy to check that $\Sigma\widetilde{X} = V(x,y,z,u)$, and

$$\pi^{-1}(\Sigma f) = V(u^2 - x^4, y, z).$$

- Let

$$X_k := \{p \in X \,||\pi^{-1}(p)| = k\}.$$

It then follows that $X_k = \emptyset$ if $k > 2$, and $X_2 = V(y,z) \backslash \{\mathbf{0}\}$, so that

$$\operatorname{supp} \mathbf{N}^\bullet_X = V(y,z) = \Sigma f.$$

A Trivial, Non-Trivial Example

- We have the short exact sequence

$$0 \to \mathbb{Q} \to H^{-2}(\pi_* \mathbf{I}^\bullet_{\widetilde{X}})_p \to H^{-1}(\mathbf{N}^\bullet_X)_p \to 0$$

  and isomorphism $H^{-1}(\pi_* \mathbf{I}^\bullet_{\widetilde{X}})_p \cong H^0(\mathbf{N}^\bullet_X)_p$.

- We can then see that verifying the stalk cohomology of $\mathbf{N}^\bullet_X$ is concentrated in degree -1 is equivalent to verifying $H^{-1}(\pi_* \mathbf{I}^\bullet_{\widetilde{X}})_p = 0$ for all $p \in X$.

- Conversely, since $\mathbf{I}^\bullet_{\widetilde{X}|_{\widetilde{X}\setminus\Sigma\widetilde{X}}} \cong \mathbb{Q}^\bullet_{\widetilde{X}\setminus\Sigma\widetilde{X}}[2]$, $\widetilde{X}$ is a $\mathbb{Q}$-homology manifold if the stalk cohomology of $\mathbf{I}^\bullet_{\widetilde{X}}$ at $\mathbf{0} \in \widetilde{X}$ is non-zero only in degree -2, where it is of dimension one.

A Trivial, Non-Trivial Example

- At $\mathbf{0} \in \widetilde{X}$, we find

$$H^{-2}(\mathbf{I}^\bullet_{\widetilde{X}})_\mathbf{0} \cong \mathbb{H}^{-2}(K_{\widetilde{X},\mathbf{0}}; \mathbf{I}^\bullet_{\widetilde{X}}) \cong H^1(K_{\widetilde{X},\mathbf{0}}; \mathbb{Q}),$$

where $K_{\widetilde{X},\mathbf{0}} = \widetilde{X} \cap S_\epsilon$ (for $0 < \epsilon \ll 1$) is the **real link** of $\widetilde{X}$ at $\mathbf{0}$.

- Since $\widetilde{X}$ has an isolated singularity at the origin, the link $K_{\widetilde{X},\mathbf{0}}$ is a compact, orientable, smooth manifold of real dimension 3. Hence, $H^0(K_{\widetilde{X},\mathbf{0}}; \mathbb{Q}) \cong H^3(K_{\widetilde{X},\mathbf{0}}; \mathbb{Q}) \cong \mathbb{Q}$.

A Trivial, Non-Trivial Example

- The standard parameterization of the twisted cubic in $\mathbb{P}^3$ lifts to a parameterization of $\widetilde{X}$, which is isomorphic to the affine cone over the twisted cubic. This parameterization is 3-to-1, from which it follows (with some work) that we have a 3-to-1 cover of $K_{\widetilde{X},\mathbf{0}}$ by the 3-sphere in $\mathbb{C}^2$. Hence,

$$H_1(K_{\widetilde{X},\mathbf{0}};\mathbb{Z}) \cong H^2(K_{\widetilde{X},\mathbf{0}};\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}.$$

- By the Universal Coefficient Theorem and Poincaré Duality,

$$H^2(K_{\widetilde{X},\mathbf{0}};\mathbb{Q}) \cong H^1(K_{\widetilde{X},\mathbf{0}};\mathbb{Q}) = 0,$$

so that $\widetilde{X}$ is a $\mathbb{Q}$-homology manifold.

Generalized Conserved Quantity Theorem

- Examining the proof of the Conserved Quantity Theorem, it is now clear that one can replace our finite, generically one-to-one maps $\pi_0 : (\mathcal{W}, S) \to (V(f_0), \mathbf{0})$ (where $\mathcal{W} \subseteq \mathbb{C}^n$ is smooth) with the normalization $\pi_0 : (\widetilde{V}(f_0), S) \to (V(f_0), \mathbf{0})$ of $V(f_0)$, provided that $\widetilde{V}(f_0)$ is a $\mathbb{Q}$-homology manifold.

- We call X a **parameterized space** if $\mathbb{Z}_X^\bullet[n]$ is perverse and the normalization $\widetilde{X}$ of X is smooth (using $\mathbb{Z}$-coefficients). Similarly, if we use $\mathbb{Q}$-coefficients, we say X is a parameterized space if $\mathbb{Q}_X^\bullet[n]$ is perverse and the normalization $\widetilde{X}$ is a $\mathbb{Q}$-homology manifold.

Hodge Theory Interlude: When is $\mathbf{N}^\bullet_X$ Semi-Simple?

- Over $\mathbb{Q}$, the category of perverse sheaves on X is locally Artinian, with **simple objects** corresponding to intersection cohomology complexes $\mathbf{I}^\bullet_Z(\mathcal{L})$ supported on irreducible subvarieties Z of X and coefficients in irreducible local systems $\mathcal{L}$.
- Then, from fundamental short exact sequence

$$0 \to \mathbf{N}^\bullet_X \to \mathbb{Q}^\bullet_X[n] \to \mathbf{I}^\bullet_X \to 0,$$

we note that $\mathbf{I}^\bullet_X$ is a semi-simple perverse sheaf. It is then a natural question to ask:

Question

Is $\mathbf{N}^\bullet_X$ ever semi-simple, so that $\mathbb{Q}^\bullet_X[n]$ is an extension of semi-simples?

Strategy: The Weight Filtration on $\mathbb{Q}_X^\bullet[n]$

- Via Morihiko Saito's theory of mixed Hodge modules [15], the natural quotient morphism $\mathbb{Q}_X^\bullet[n] \to \mathbf{I}_X^\bullet$ induces an isomorphism

$$\mathrm{Gr}_n^W \mathbb{Q}_X^\bullet[n] \cong \mathbf{I}_X^\bullet,$$

  where $\mathrm{Gr}_n^W \mathbb{Q}_X^\bullet[n]$ is the n-th graded piece of the **weight filtration** on $\mathbb{Q}_X^\bullet[n]$, considered as a mixed Hodge module on X.
- Consequently, the fundamental short exact sequence identifies the comparison complex $\mathbf{N}_X^\bullet$ with $W_{n-1}\mathbb{Q}_X^\bullet[n]$, and $\mathbf{N}_X^\bullet$ inherits the structure of a mixed Hodge module with weight filtration

$$W_k \mathbf{N}_X^\bullet = W_k \mathbb{Q}_X^\bullet[n] \quad \text{for } k \leq n-1.$$

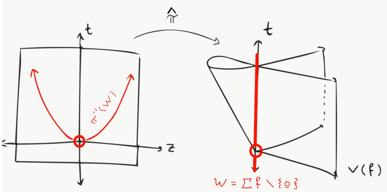
- Answering our question is then equivalent to understanding this weight filtration on $\mathbf{N}_X^\bullet$.

The Set-Up

- Let ΣX denote the singular locus of X, and let $i : \Sigma X \hookrightarrow X$. We can then find a smooth, Zariski open dense subset $\mathcal{W} \subseteq \Sigma X$ over which the normalization map restricts to a covering projection [3]

$$\hat{\pi} : \pi^{-1}(\mathcal{W}) \to \mathcal{W} \subseteq \Sigma X.$$

- Let $l : \mathcal{W} \hookrightarrow \Sigma X$ and $m : \Sigma X \backslash \mathcal{W} \hookrightarrow \Sigma X$ denote the respective open and closed inclusion maps. Let $\hat{m} := i \circ m$, $\hat{l} := i \circ l$.



Theorem (H.,2018 [6])

Suppose X is a parameterized space. Then, there is an isomorphism $\mathrm{Gr}^W_{n-1}\, i^*\mathbf{N}^\bullet_X \cong \mathbf{I}^\bullet_{\Sigma X}(\hat{I}^*\mathbf{N}^\bullet_X)$, *so that the short exact sequence of perverse sheaves on X*

$$0 \to m_* {}^pH^0(m^! i^*\mathbf{N}^\bullet_X) \to i^*\mathbf{N}^\bullet_X \to \mathbf{I}^\bullet_{\Sigma X}(\hat{I}^*\mathbf{N}^\bullet_X) \to 0$$

identifies $W_{n-2} i^*\mathbf{N}^\bullet_X \cong m_* {}^pH^0(m^! i^*\mathbf{N}^\bullet_X)$. *Here,* $\mathbf{I}^\bullet_{\Sigma X}(\hat{I}^*\mathbf{N}^\bullet_X)$ *denotes the intermediate extension of the perverse sheaf* $\hat{I}^*\mathbf{N}^\bullet_X$ *to all of* ΣX, *and* ${}^pH^0(-)$ *denotes the* 0*-th perverse cohomology functor.*

Corollary

Suppose X is a parameterized space. Then, there are isomorphisms

$$\mathrm{Gr}^W_{n-1}\, \mathbb{Q}^\bullet_X[n] \cong \mathrm{Gr}^W_{n-1}\, \mathbf{N}^\bullet_X \cong i_*\mathbf{I}^\bullet_{\Sigma X}(\hat{I}^*\mathbf{N}^\bullet_X).$$

Sketch of Theorem

- We show that there exists a short exact sequence of perverse sheaves (underlying MHM)

$$0 \to m_* {}^pH^0(m^! i^* \mathbf{N}_X^\bullet) \to i^* \mathbf{N}_X^\bullet \to \mathbf{I}_{\Sigma X}^\bullet(\hat{l}^* \mathbf{N}_X^\bullet) \to 0,$$

  provided $\hat{m}^*[-1]\mathbf{N}_X^\bullet \in {}^pD^{\leq 0}(\Sigma X \backslash \mathcal{W})$, called the **enhanced support condition** along $\Sigma \backslash \mathcal{W}$.

- Next, we show $\mathbf{I}_{\Sigma X}^\bullet(\hat{l}^* \mathbf{N}_X^\bullet)$ underlies a **polarizable Hodge module** of weight $n-1$.

- Finally, we prove $\mathrm{Gr}_{n-1}^W\, i^* \mathbf{N}_X^\bullet \cong \mathbf{I}_{\Sigma X}^\bullet(\hat{l}^* \mathbf{N}_X^\bullet)$ by using the decomposition by strict support property on $\mathrm{Gr}_{n-1}^W\, i^* \mathbf{N}_X^\bullet$.

The Parameterized Surface Case

- Suppose $X = V(f)$ is a parameterized surface in $\mathbb{C}^3$; we want to compute $W_0\mathbb{Q}^\bullet_{V(f)}[2]$ using the isomorphism

$$W_0\mathbb{Q}^\bullet_{V(f)}[2] = W_0\mathbf{N}^\bullet_{V(f)} \cong \hat{m}_* {}^pH^0(m^! i^* \mathbf{N}^\bullet_{V(f)}).$$

- The main tool we use is the following: if $\dim_{\mathbf{0}} \Sigma f = 1$, then $\Sigma f \backslash \mathcal{W}$ is zero-dimensional (or empty), and perverse cohomology on a zero-dimensional space is just ordinary cohomology.

$$\hat{m}_* {}^pH^0(m^! i^* \mathbf{N}^\bullet_{V(f)}) \cong \mathbb{H}^0(\Sigma f, \Sigma f \backslash \{\mathbf{0}\}; i^* \mathbf{N}^\bullet_{V(f)}).$$

Theorem (H.,2018 [6])

Suppose $V(f)$ is a parameterized surface in $\mathbb{C}^3$. Then,

$$W_0\mathbb{Q}^\bullet_{V(f)}[2] \cong V^\bullet_{\{\mathbf{0}\}}$$

is a perverse sheaf concentrated on a single point, i.e., a finite-dimensional $\mathbb{Q}$-vector space, of dimension

$$\dim V = 1 - |\pi^{-1}(\mathbf{0})| + \sum_C \dim \ker\{\mathrm{id} - h_C\},$$

where $\{C\}$ is the collection of irreducible components of Σf at $\mathbf{0}$, and for each component C, h_C is the (internal) monodromy operator on the local system $H^{-1}(\mathbf{N}^\bullet_{V(f)})_{|_{C\setminus\{\mathbf{0}\}}}$. Note that $|\pi^{-1}(\mathbf{0})|$ is, of course, equal to the number of irreducible components of $V(f)$ at $\mathbf{0}$.

Question

For parameterized surfaces $V(f) \subseteq \mathbb{C}^3$, what does the vanishing of $W_0\mathbf{N}^\bullet_{V(f)}$ imply about the topology of $V(f)$?

Examples

- Let $f(x,y,t) = y^2 - x^3 - tx^2$, so that $V(f)$ is the Whitney umbrella. Then, $\Sigma f = V(x,y)$, and $V(f)$ has (smooth) normalization given by $\pi(t,u) = (u^2 - t, u(u^2 - t), t)$. Then, it is easy to see that the internal monodromy operator h_C along the component $V(x,y)$ is multiplication by -1, so $\ker\{\mathrm{id} - h_C\} = 0$. Hence,

$$W_0 \mathbb{Q}^\bullet_{V(f)}[2] = 0.$$

- Let $f(x,y,z) = xz^2 - y^3$, so that $\Sigma f = V(y,z)$. Then, the normalization $\widetilde{V(f)}$ is equal to

$$\widetilde{V(f)} = V(u^2 - xy, uy - xz, uz - y^2) \subseteq \mathbb{C}^4.$$

The internal monodromy operator h_C on $H^{-1}(\mathbf{N}^\bullet_{V(f)})_{|_{V(y,z)\setminus\{\mathbf{0}\}}}$ is trivial, so $\ker\{\mathrm{id} - h_C\} \cong \mathbb{Q}$. Thus,

$$W_0 \mathbb{Q}^\bullet_{V(f)}[2] \cong \mathbb{Q}^\bullet_{\{\mathbf{0}\}}.$$

Future Direction: $\mathbf{N}^\bullet_{V(f)}$ and Milnor Monodromy

- For (reduced) hypersurfaces $V(f)$, $\mathbf{N}^\bullet_{V(f)} \cong \ker\{\mathrm{id} - \widetilde{T}_f\}$ by a result of Massey [9], where $\widetilde{T}_f$ is the Milnor monodromy action on the vanishing cycles $\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]$.

- With a bit more work, one ca,n deduce from Massey's result that one has an isomorphism of mixed Hodge modules

$$\mathbf{N}^\bullet_{V(f)} \cong \ker N(1),$$

where $N = \frac{1}{2\pi i}\log T_u$, T_u is the unipotent part of the monodromy operator $\widetilde{T}_f$, and (1) denotes the Tate twist operator.

- Hence, $W_k\mathbf{N}^\bullet_{V(f)} \cong W_k \ker N(1) \cong W_{k+2}\ker N$ for all $k \leq n-1$.

Future Direction: The Algebraic Setting and Saito's Work

- When X is a reduced complex algebraic variety of pure dimension n, [16] has recently shown that

$$W_0 H^1(X; \mathbb{Q}) \cong \mathrm{Coker}\{H^0(\widetilde{X}; \mathbb{Q}) \to \mathbb{H}^{-n+1}(X; \mathbf{N}_X^\bullet)\},$$

  where $W_0 H^1(X; \mathbb{Q})$ denotes the weight zero part of the cohomology $H^1(X; \mathbb{Q})$, considered as a mixed Hodge structure, and $\widetilde{X}$ is the normalization of X.

- Saito additionally shows that $W_0 H^1(X; \mathbb{Q})$ is a topological invariant associated to X, and has dimension

$$\dim_{\mathbb{Q}} W_0 H^1(X; \mathbb{Q}) = \dim_{\mathbb{Q}} \mathbb{H}^{-n+1}(X; \mathbf{N}_X^\bullet) - b_0(\widetilde{X}) + b_0(X).$$

  where $b_0(\widetilde{X})$ and $b_0(X)$ are the 0-th Betti numbers of $\widetilde{X}$ and X, respectively.

Bobadilla's Conjecture

- My second project concerns a conjecture by Javier Fernández de Bobadilla [1] related to Lê's Conjecture.
- Suppose $f : (\mathbb{C}^{n+1}, \mathbf{0}) \to (\mathbb{C}, 0)$ is a complex analytic function germ with $\dim_{\mathbf{0}} \Sigma f = 1$, and denote by $F_{f,\mathbf{0}}$ the Milnor fiber of f at the origin.

Conjecture (Fernández de Bobadilla)

Suppose that the reduced integral cohomology $\widetilde{H}^k(F_{f,\mathbf{0}}; \mathbb{Z})$ is non-zero only in degree $(n-1)$, and that

$$\widetilde{H}^{n-1}(F_{f,\mathbf{0}}; \mathbb{Z}) \cong \bigoplus_C \mathbb{Z}^{\overset{\circ}{\mu}_C}$$

where the sum is over all irreducible components C of Σf at $\mathbf{0}$, and $\overset{\circ}{\mu}_C$ denotes the generic transverse Milnor number of f along C. Then, in fact, Σf has a single irreducible component, which is smooth.

Massey and H. made some progress on this conjecture [8], proving that it does hold in some special cases; these are the first known positive results toward the conjecture. In particular, we prove:

Theorem (H., Massey [8])

Bobadilla's Conjecture holds in the following cases.

1. *We prove an induction-like result for when f is a sum of two analytic functions defined on disjoint sets of variables.*
2. *We prove the result for the case when the relative polar curve Γ^1_{f,z_0} is defined by a single equation inside the relative polar surface $\Gamma^2_{f,\mathbf{z}}$. In particular, the conjecture is true for any non-reduced plane curve singularity.*

Future Direction: Lê's Conjecture

Parameterized Surfaces in $\mathbb{C}^3$ are the subject of a long-standing conjecture by Lê Dũng Tráng [13], in the vein of classical equisingularity problems of Mumford [14] and Zariski, and is related to Bobadilla's Conjecture [2].

Conjecture (Lê)

Suppose $(V(f), \mathbf{0}) \subseteq (\mathbb{C}^3, \mathbf{0})$ *is a reduced hypersurface with* $\dim_{\mathbf{0}} \Sigma f = 1$*, for which the normalization of* $V(f)$ *is a bijection. Then, in fact,* $V(f)$ *is the total space of an equisingular deformation of plane curve singularities.*

- We note that the assumption of the normalization of $V(f)$ being a bijection is equivalent to $\mathbf{N}^\bullet_{V(f)} = 0$, and that the conjecture is equivalent to the vanishing $\phi_L[-1]\mathbb{Z}^\bullet_{V(f)}[2] = 0$ for generic linear forms L on $\mathbb{C}^3$.
- $\mathbf{N}^\bullet_{V(f)} = 0$ also clearly implies $W_0\mathbf{N}^\bullet_{V(f)} = 0$.

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Thank You!

Slides up at https://brainhelper.wordpress.com/

Appendix: IPA-Deformations [10]

In order to compute the characteristic polar multiplicities and Lê numbers, we need to choose linear forms that "cut down" the support in a certain way. We now give several equivalent conditions for this "cutting" procedure.

Proposition

*f **is a deformation of** $f_{|_{V(L)}}$ **with isolated polar activity at 0** if any of the equivalent hold:*

1. $\dim_{\mathbf{0}} \Gamma^1_{f,L} \cap V(L) \leq 0$.
2. $\dim_{(\mathbf{0}, d_{\mathbf{0}}L)} \operatorname{im} dL \cap (f \circ \pi)^{-1}(0) \cap \overline{T^*_f \mathcal{U}}$, *where* $\pi : T^*\mathcal{U} \to \mathcal{U}$ *is the canonical projection map.*
3. $\dim_{(\mathbf{0}, d_{\mathbf{0}}L)} SS(\psi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]) \cap \operatorname{im} dL \leq 0$.
4. $\dim_{(\mathbf{0}, d_{\mathbf{0}}L)} SS(\mathbb{Z}^\bullet_{V(f)}[n]) \cap \operatorname{im} dL \leq 0$.
5. $\dim_{\mathbf{0}} \operatorname{supp} \phi_L[-1]\mathbb{Z}^\bullet_{V(f)}[n] \leq 0$.

Appendix: Characteristic Polar Multiplicities [11]

- Given a perverse sheaf $\mathbf{P}^\bullet$ on a complex analytic subset X of $\mathbb{C}^N$, and choice of "nice" tuple of linear forms $\mathbf{z} = (z_0, \ldots, z_s)$ on $\mathbb{C}^N$ (where $\dim \operatorname{supp} \mathbf{P}^\bullet = s$), the **characteristic polar multiplicities of $\mathbf{P}^\bullet$ with respect to $\mathbf{z}$** at a point $p \in X$ are the non-negative integers

  $$\lambda^i_{\mathbf{P}^\bullet,\mathbf{z}}(p) = \operatorname{rank}_{\mathbb{Z}} H^0(\phi_{z_i - z_i(p)}[-1]\psi_{z_{i-1}-z_{i-1}(p)}[-1] \cdots \psi_{z_0 - z_0(p)}[-1]\mathbf{P}^\bullet)_p,$$

  for $0 \leq i \leq s$.
- Such numbers exist more generally for objects of $D^b_{\mathbb{C}-c}(X)$, but they are slightly more cumbersome to define (and no longer need to be non-negative!)
- Why are these useful? For all $p \in X$, one has

  $$\chi(\mathbf{P}^\bullet)_p = \sum_{i=0}^{s} (-1)^i \lambda^i_{\mathbf{P}^\bullet,\mathbf{z}}(p).$$

Appendix: $\mathbf{N}_X^\bullet$ and Small Maps Example

$f(x,y,z,w) = xw - yz$, $X = V(f) \subseteq \mathbb{C}^4$, and set $Y = V(xt_0 + yt_1, zt_0 + wt_1) \subseteq \mathbb{C}^4 \times \mathbb{P}^1$. Then, the projection $\pi : Y \to X$ is a small map, where Y is smooth, and one-to-one away from $\mathbf{0}$ in X. We then find:

$$H^k(\mathbf{N}_X^\bullet)_p = 0$$

for all $p \neq \mathbf{0}$ and all $k \in \mathbb{Z}$. At $\mathbf{0}$, we find:

$$H^k(\mathbf{N}_X^\bullet)_{\mathbf{0}} \cong \widetilde{H}^{k+2}(\mathbb{P}^1;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0 \\ 0, & \text{else} \end{cases}$$