Hypersurface Normalizations and Numerical Invariants

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The Fundamental Short Exact Sequence

Suppose X is a purely n-dimensional local complete intersection inside some open neighborhood of the origin in $\mathbb{C}^N$. Then, the shifted constant sheaf $\mathbb{Z}_X^\bullet[n]$ is perverse, and there is a canonical morphism $\mathbb{Z}_X^\bullet[n] \xrightarrow{c} \mathbf{I}_X^\bullet$, where $\mathbf{I}_X^\bullet$ is the intersection cohomology complex on X with constant $\mathbb{Z}$ coefficients. Since $\mathbf{I}_X^\bullet$ is also the intermediate extension of the constant sheaf on X_{reg}, it has no non-trivial sub-perverse sheaves or quotient-perverse sheaves with support contained in ΣX. Therefore, since our morphism induces an isomorphism when restricted to X_{reg}, its cokernel must be zero, i.e., the morphism c is a surjection.

We let $\mathbf{N}_X^\bullet$ be the kernel of the morphism c, so that there is a short exact sequence of perverse sheaves

$$0 \to \mathbf{N}_X^\bullet \to \mathbb{Z}_X^\bullet[n] \to \mathbf{I}_X^\bullet \tag{1}$$

on X. As $\mathbb{Z}_X^\bullet[n]$ and $\mathbf{I}_X^\bullet$ are, essentially, the two fundamental perverse sheaves on the LCI X, we refer to 1 as the **fundamental short exact sequence of the LCI**. This short exact sequence, and the perverse sheaf $\mathbf{N}_X^\bullet$ in particular, have examined recently in several papers by the author and D. Massey in the case where the normalization of X is smooth [5] and [3], and where the normalization is a rational homology manifold [4]. In these papers, we refer to $\mathbf{N}_X^\bullet$ as the **multiple-point complex** of the normalization, as it naturally encodes the data about the image multiple-points of the normalization.

Relationship with the Vanishing Cycles [H., 2017]

In the case where $X = V(f)$ is a hypersurface in some open neighborhood $\mathcal{U}$ of the origin in $\mathbb{C}^{n+1}$, we prove in [3] that a strong relationship holds between the **characteristic polar multiplicities** of $\mathbf{N}_X^\bullet$ and the Lê numbers of the function f. This same result holds for hypersurface normalizations that are $\mathbb{Q}$-homology manifolds. More precisely, the exact same proof yields:

Theorem 1 (H., 2017 [3])**.** *Suppose that $\widetilde{X}$ is a $\mathbb{Q}$-homology manifold, and $\pi : (\widetilde{X} \times \mathbb{C}, \{0\} \times S) \to (\mathcal{U}, \mathbf{0})$ is a one-parameter unfolding with parameter t, with $\operatorname{im} \pi = X = V(f)$ for some $f \in \mathcal{O}_{\mathcal{U},\mathbf{0}}$. Suppose further that $\boldsymbol{z} = (z_1, \cdots, z_n)$ is chosen such that $\boldsymbol{z}$ is an IPA-tuple for $f_0 = f_{|_{V(t)}}$ at $\mathbf{0}$. Then, if we let $\boldsymbol{N}_{X_{t_0}}^\bullet = \boldsymbol{N}_{X|_{V(t-t_0)}}^\bullet[-1]$, the following formulas hold for the Lê numbers of f_0 with respect to $\boldsymbol{z}$ at $\mathbf{0}$: for $0 < |t_0| \ll \epsilon \ll 1$, for $0 \leq i \leq n-2$,*

$$\lambda^i_{f_0,\boldsymbol{z}}(\mathbf{0}) + \lambda^i_{\boldsymbol{N}_{X_0}^\bullet,\boldsymbol{z}}(\mathbf{0}) = \sum_{p \in B_\epsilon \cap V(t-t_0, z_1, z_2, \cdots, z_i)} \left(\lambda^i_{f_{t_0},\boldsymbol{z}}(p) + \lambda^i_{\boldsymbol{N}_{X_{t_0}}^\bullet,\boldsymbol{z}}(p) \right)$$

Since the Lê numbers of f are the same as the characteristic polar multiplicities of the vanishing cycles $\phi_f[-1]\mathbb{Z}_{\mathcal{U}}^\bullet[n+1]$, **this result suggests a conservation of number property with the characteristic polar multiplicities of the comparison complex and the vanishing cycles.**

For a precise definition of characteristic polar multiplicities, see [6]; for deformations with isolated polar activity (IPA-deformations and IPA-tuples), see [7].

Relationship with the Monodromy on Vanishing Cycles [Massey, 2018]

If one examines the short exact sequence (1) in the case where X is a hypersurface $V(f)$, Massey has recently shown in [9] that there is an isomorphism

$$\mathbf{N}_{V(f)}^\bullet \cong \ker\{\mathrm{Id} - \widetilde{T}_f\},$$

where $\widetilde{T}_f$ is the monodromy action on the vanishing cycles $\phi_f[-1]\mathbb{Z}_{\mathcal{U}}^\bullet[d+1]$, and the kernel takes place in the category of perverse sheaves on $V(f)$. In this context, Massey refers to $\mathbf{N}_X^\bullet$ as the **comparison complex** on $V(f)$.

Consider the unipotent vanishing cycles $\phi_{f,1}[-1]\mathbb{Q}_{\mathcal{U}}^\bullet[n+1]$ as a **mixed Hodge module** with monodromy weight filtration induced by

$$N = \log T_u : \phi_{f,1}[-1]\mathbb{Q}_{\mathcal{U}}^\bullet[n+1] \to \phi_{f,1}[-1]\mathbb{Q}_{\mathcal{U}}^\bullet[n+1](-1),$$

where (-1) denotes the Tate twist operator, and T_u is the unipotent part of the monodromy operator $\widetilde{T}_f$. Then, as mixed Hodge modules, one has the isomorphism

$$\mathbf{N}_{V(f)}^\bullet \cong \ker N(1).$$

Relationship with $\mathbb{Q}$-Homology Manifolds [H., 2018]

Looking at the short exact sequence (1), one notices immediately that $\mathbb{Q}_X^\bullet[n] \cong \mathbf{I}_X^\bullet$ if and only if $\mathbf{N}_X^\bullet = 0$; that is, the LCI X is a rational homology manifold (or, a **$\mathbb{Q}$-homology manifold**) precisely when the complex $\mathbf{N}_X^\bullet$ vanishes. It is then natural to ask that, given the normalization $\pi : Y \to X$ and the resulting fundamental short exact sequence

$$0 \to \mathbf{N}_X^\bullet \to \mathbb{Q}_X^\bullet[n] \to \pi_* \mathbf{I}_Y^\bullet \to 0,$$

is there a similar result relating $\mathbf{N}_X^\bullet$ to whether or not Y is a $\mathbb{Q}$-homology manifold?

Theorem 2 (H., 2018 [4])**.** *Y is a $\mathbb{Q}$-homology manifold if and only if $\mathbf{N}_X^\bullet$ has stalk cohomology concentrated in degree $-n+1$; i.e., for all $p \in X$, $H^k(\mathbf{N}_X^\bullet)_p$ is non-zero only possibly when $k = -n+1$.*

$\mathbf{N}_X^\bullet$ as a Mixed Hodge Module [H., 2018]

By shrinking the open neighborhood $\mathcal{U}$ if necessary, $\mathbb{Q}_X^\bullet[n]$ underlies a graded-polarizable mixed Hodge module of weight $\leq n$. By Saito's theory of (graded polarizable) mixed Hodge modules in the local complex analytic context [10], the quotient morphism $\mathbb{Q}_X^\bullet[n] \to \mathbf{I}_X^\bullet$ induces an isomorphism

$$\mathrm{Gr}_n^W \mathbb{Q}_X^\bullet[n] \xrightarrow{\sim} \mathbf{I}_X^\bullet;$$

consequently, the short exact sequence (1) identifies the comparison complex $\mathbf{N}_X^\bullet$ with $W_{n-1}\mathbb{Q}_X^\bullet[n]$. This then endows $\mathbf{N}_X^\bullet$ with the structure of a mixed Hodge module of weight $\leq n-1$ with weight filtration $W_k\mathbf{N}_X^\bullet = W_k\mathbb{Q}_X^\bullet[n]$ for $k \leq n-1$.

Letting ΣX denote the singular locus of X, and let $i : \Sigma X \hookrightarrow X$. We can then find a smooth, Zariski open dense subset $\mathcal{W} \subseteq \Sigma X$ over which the normalization map restricts to a covering projection $\hat{\pi} : \pi^{-1}(\mathcal{W}) \to \mathcal{W} \subseteq \Sigma X$ (see Section 6.2, [1]). Let $l : \mathcal{W} \hookrightarrow \Sigma X$ and $m : \Sigma X \backslash \mathcal{W} \hookrightarrow \Sigma X$ denote the respective open and closed inclusion maps. Let $\hat{m} := i \circ m$, $\hat{l} := i \circ l$. Note that $\dim_{\mathbf{0}} \Sigma X \backslash \mathcal{W} \leq n-2$, as it is the complement of a Zariski open set.

Theorem 3 (H, 2018 [2])**.** *Suppose the normalization of X is a $\mathbb{Q}$-homology manifold. Then, there is an isomorphism $\mathrm{Gr}_{n-1}^W i^*\boldsymbol{N}_X^\bullet \cong \boldsymbol{I}_{\Sigma X}^\bullet(\hat{l}^*\boldsymbol{N}_X^\bullet)$, so that the short exact sequence of perverse sheaves on X*

$$0 \to m_*{}^pH^0(m^! i^*\boldsymbol{N}_X^\bullet) \to i^*\boldsymbol{N}_X^\bullet \to \boldsymbol{I}_{\Sigma X}^\bullet(\hat{l}^*\boldsymbol{N}_X^\bullet) \to 0$$

identifies $W_{n-2}i^*\boldsymbol{N}_X^\bullet \cong m_*{}^pH^0(m^! i^*\boldsymbol{N}_X^\bullet)$. Here, $\boldsymbol{I}_{\Sigma X}^\bullet(\hat{l}^*\boldsymbol{N}_X^\bullet)$ denotes the intermediate extension of the perverse sheaf $\hat{l}^*\boldsymbol{N}_X^\bullet$ to all of ΣX, and ${}^pH^0(-)$ denotes the 0-th perverse cohomology functor.

Since the map $i : \Sigma X \hookrightarrow X$ is a closed inclusion, it preserves weights. Moreover, the support of $\mathbf{N}_X^\bullet$ is contained in the singular locus ΣX, and so $i_*i^*\mathbf{N}_X^\bullet \cong \mathbf{N}_X^\bullet$. Consequently, we have the following.

Corollary 4 (H, 2018 [2])**.** *Suppose the normalization of X is a $\mathbb{Q}$-homology manifold. Then, there are isomorphisms*

$$\mathrm{Gr}_{n-1}^W \mathbb{Q}_X^\bullet[n] \cong \mathrm{Gr}_{n-1}^W \boldsymbol{N}_X^\bullet \cong i_*\boldsymbol{I}_{\Sigma X}^\bullet(l^*\boldsymbol{N}_X^\bullet),$$

and

$$W_{n-2}\mathbb{Q}_X^\bullet[n] \cong W_{n-2}\boldsymbol{N}_X^\bullet \cong \hat{m}_*{}^pH^0(m^! i^*\boldsymbol{N}_X^\bullet) \cong \hat{m}_* \ker\{\phi_g[-1]i^*\boldsymbol{N}_X^\bullet \xrightarrow{\mathrm{var}} \psi_g[-1]i^*\boldsymbol{N}_X^\bullet\},$$

where $g : (\Sigma X, \mathbf{0}) \to (\mathbb{C}, 0)$ is any non-constant analytic function for which $V(g)$ contains $\Sigma X \backslash \mathcal{U}$, but not any irreducible component of ΣX.

In the case where $X = V(f)$ is a surface in $\mathbb{C}^3$, we explicitly compute $W_0\mathbb{Q}_{V(f)}^\bullet[2]$; the vanishing of this perverse sheaf places strong constraints on the topology of the singular set Σf of $V(f)$.

Theorem 5 (H,2018 [2])**.** *If $V(f)$ is a surface in $\mathbb{C}^3$ whose normalization is a $\mathbb{Q}$-homology manifold, and $\dim_{\mathbf{0}} \Sigma f = 1$, then*

$$W_0\mathbb{Q}_{V(f)}^\bullet[2] \cong V_{\{\mathbf{0}\}}^\bullet$$

is a perverse sheaf concentrated on a single point, i.e., a finite-dimensional $\mathbb{Q}$-vector space, of dimension

$$\dim_{\mathbb{Q}} V = 1 - |\pi^{-1}(\mathbf{0})| + \sum_C \dim_{\mathbb{Q}} \ker\{\mathrm{Id} - h_C\},$$

where $\{C\}$ is the collection of irreducible components of Σf at $\mathbf{0}$, and for each C, h_C is the (internal) monodromy operator on the local system $H^{-1}(\boldsymbol{N}_{V(f)}^\bullet)_{|_{C\backslash\{\mathbf{0}\}}}$. Note that $|\pi^{-1}(\mathbf{0})|$ is, of course, equal to the number of irreducible components of $V(f)$ at $\mathbf{0}$.

Future Directions

Question 1: Is there a result analogous to $\mathbf{N}_X^\bullet \cong \mathrm{Ker}\{\mathrm{Id} - \widetilde{T}_f\}$ in the general case of an LCI? **How is the comparison complex related (if at all) to the monodromies of the functions defining an LCI?**

Question 2: One notes that the dimension of the vector space V is very similar to the beta invariant of Massey [8]. Does its vanishing have a similar geometric significance to the vanishing of β_f?

It is possible for $V = 0$; this happens, e.g., for the Whitney umbrella $V(y^2 - x^3 - zx^2)$ for which Σf is smooth at the origin, but this is not a sufficient condition. Indeed, the critical locus of $V(xz^2 - y^3)$ is smooth at $\mathbf{0}$, but $V = \mathbb{Q}$.

However, we may distinguish these examples by noting that, for generic linear forms L, the normalization map $\pi : Y \to V(f)$ is a **simultaneous normalization** of the the family

$$\pi_\xi : Y \cap (L \circ \pi)^{-1}(\xi) \to V(f, L - \xi)$$

for all $\xi \in \mathbb{C}$ small in the case of the Whitney umbrella, but **not** for the surface $V(xz^2 - y^3)$. Is this true in general? This would make the perverse sheaf $W_0\mathbb{Q}_{V(f)}^\bullet[2]$ very relevant to **Lê's Conjecture**.

Question 3: When X is a reduced **complex algebraic variety** of pure dimension n, Morihiko Saito [11] has recently shown that

$$W_0H^1(X;\mathbb{Q}) \cong \mathrm{Coker}\{H^0(Y;\mathbb{Q}) \to \mathbb{H}^{-n+1}(X;\mathbf{N}_X^\bullet)\},$$

where $W_0H^1(X;\mathbb{Q})$ denotes the weight zero part of the cohomology $H^1(X;\mathbb{Q})$, considered as a mixed Hodge Structure, and Y is the normalization of X. **How much of the relationship between $\mathbf{N}_X^\bullet$ and the Vanishing Cycles (and their monodromy actions) persists in this general setting of arbitrary reduced complex algebraic varieties? What is the extent of the link with Mixed Hodge Modules?**

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