RESEARCH STATEMENT

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Background

The main focus of my research is the local topology of complex analytic spaces with arbitrary singularities. Largely, this research is concerned with the use of perverse sheaves and derived category techniques in extending classical results for isolated singularities in affine or projective space to arbitrarily singular complex analytic spaces.

The main object of study that threads together my research is a perverse sheaf called the **comparison complex**, first defined by myself and David Massey in [9] (where we originally referred to it as the **multiple-point complex**, for reasons that will become clear in Subsection 1.1), and subsequently studied in several papers by myself in [6], [7], and Massey in [12]. This perverse sheaf, denoted $\mathbf{N}^\bullet_X$, is defined on any pure-dimensional (locally reduced) complex analytic space X for which the constant sheaf $\mathbb{Z}^\bullet_X[\dim X]$ is perverse.

In a sentence, I am interested in studying the numerous connections between the comparison complex and the vanishing cycles on affine space, and more generally on spaces X for which the complex $\mathbb{Z}^\bullet_X[n]$ is perverse (e.g., local complete intersections). In particular, I am interested in this connection appearing in one-parameter unfoldings of hypersurface normalizations (see Section 1.2), and in a well-known conjecture of Lê (see Section 2).

General Set-Up. Let $\mathcal{U}$ be an open neighborhood of the origin in $\mathbb{C}^N$, let $X \subseteq \mathcal{U}$ be a reduced complex analytic space, containing $\mathbf{0}$, of pure dimension n, on which the (shifted) constant sheaf $\mathbb{Z}^\bullet_X[n]$ is perverse. There is then a canonical surjection of perverse sheaves $\mathbb{Z}^\bullet_X[n] \to \mathbf{I}^\bullet_X \to 0$, where $\mathbf{I}^\bullet_X$ is the intersection cohomology complex on X with constant $\mathbb{Z}$ coefficients.

Since the category of perverse sheaves on X is Abelian, there is a perverse sheaf $\mathbf{N}^\bullet_X$ on X such that

$$0 \to \mathbf{N}^\bullet_X \to \mathbb{Z}^\bullet_X[n] \to \mathbf{I}^\bullet_X \to 0 \tag{1}$$

is a short exact sequence of perverse sheaves. We call $\mathbf{N}^\bullet_X$ the **comparison complex** on X; by definition, it is supported on ΣX. On any space for which the constant sheaf is perverse, $\mathbb{Z}^\bullet_X[n]$ and $\mathbf{I}^\bullet_X$ are, in a sense, the two "fundamental" perverse sheaves that are naturally defined. It is our belief that the short exact sequence Formula 1 is the most natural object to study if one wishes to understand the local topology of X. For this reason, we refer to Formula 1 as the **fundamental short exact sequence of** X.

Massey has recently shown in [12] that, in the case where $X = V(f)$ is a hypersurface, $\mathbf{N}^\bullet_X \cong \ker\{\mathrm{id} - \widetilde{T}_f\}$, where $\widetilde{T}_f$ is the monodromy action on the vanishing cycles $\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]$, and the kernel takes place in the category of perverse sheaves on $X = V(f)$. Consequently, one has the interpretation of $\mathbf{N}^\bullet_X$ as being contained in the **unipotent part of the vanishing cycles**. It also seems that one may obtain this result in the algebraic setting (with $\mathbb{Q}$ coefficients) using the language of mixed Hodge modules [20].

Using $\mathbb{Q}$-coefficients, $\mathbf{N}^\bullet_X$ has the natural structure of a mixed Hodge module in the local analytic setting. When $\mathbb{Q}^\bullet_X[n]$ is a perverse sheaf, one may use the general machinery of M. Saito (see [20], page 325 (4.5.9)) to conclude $\mathbf{N}^\bullet_X \cong W_{n-1}\mathbb{Q}^\bullet_X[n]$. We will examine this in more detail in Section 1.3.

In the algebraic setting, M. Saito has recently shown in [21] that one may obtain the weight zero part of the cohomology group $H^1(X;\mathbb{Q})$ as the cokernel of the canonical morphism $H^0(Y;\mathbb{Q}) \to \mathbb{H}^{-n+1}(X;\mathbf{N}^\bullet_X)$, where Y is the normalization of X.

1. Project 1: Investigating the Comparison Complex

1.1. Rational Homology Manifolds, Normalizations, and the Comparison Complex.

Unless otherwise specified, we will use $\mathbb{Q}$ coefficients in this subsection. Of course, the canonical map is an isomorphism $\mathbb{Q}^\bullet_X[n] \xrightarrow{\sim} \mathbf{I}^\bullet_X$ if and only if $\mathbf{N}^\bullet_X = 0$; it is then a theorem by Borho and MacPherson [3] that the space X is a rational homology manifold (or, a $\mathbb{Q}$**-homology manifold**) precisely when the complex $\mathbf{N}_X^\bullet$ vanishes (see [3],[13]).

Let $\pi : Y \to X$ be the normalization of X. Then, π is a **small map**, in the sense of Goresky and MacPherson [5], so that $\pi_*\mathbf{I}_Y^\bullet \cong \mathbf{I}_X^\bullet$, where $\mathbf{I}_Y^\bullet$ is intersection cohomology on Y with constant $\mathbb{Q}$ coefficients. Thus, the fundamental short exact sequence on X takes the form

$$0 \to \mathbf{N}_X^\bullet \to \mathbb{Q}_X^\bullet[n] \to \pi_*\mathbf{I}_Y^\bullet \to 0. \tag{2}$$

It is then natural to ask that, given this short exact sequence, is there a similar result relating $\mathbf{N}_X^\bullet$ to whether or not Y is a $\mathbb{Q}$-homology manifold?

I answer this question in the main result of [7]:

Theorem 1.1 (H. '18)**.** *Y is a $\mathbb{Q}$-homology manifold if and only if $\mathbf{N}_X^\bullet$ has stalk cohomology concentrated in degree $-n+1$; i.e., for all $p \in X$, $H^k(\mathbf{N}_X^\bullet)_p$ is possibly non-zero only when $k = -n+1$.*

When the normalization $Y \xrightarrow{\pi} X$ is a $\mathbb{Q}$-homology manifold, we can make the identification

$$m(p) := \dim_{\mathbb{Q}} H^{-n+1}(\mathbf{N}_X^\bullet)_p = |\pi^{-1}(p)| - 1. \tag{3}$$

Consequently, we conclude that the support of $\mathbf{N}_X^\bullet$ is none other than the **image multiple-point set** of the morphism π (also known as the non-unibranch locus of X), which we denote by D; precisely, we have $D := \overline{\{p \in X \mid |\pi^{-1}(p)| > 1\}}$. For this reason, we have referred to the perverse sheaf $\mathbf{N}_X^\bullet$ in this context as the **multiple-point complex** of X (or, of the morphism π, as we do in [6] and [9]). It is immediate from the fundamental short exact sequence that one always has the inclusion $D \subseteq \Sigma X$ (this inclusion can be strict whenever the normalization of X is a bijection, e.g., the cusp $y^2 = x^3$ in $\mathbb{C}^2$).

Our original motivation for studying $\mathbf{N}_X^\bullet$ was the case where the normalization of X is smooth (where we used $\mathbb{Z}$ coefficients), and we referred to such spaces as **parameterized** by the morphism $\pi : Y \to X$. In particular, we were interested in studying the complex link of such spaces. If $X = V(f)$ is a hypersurface, and L is a generic linear form on $\mathcal{U}$, then the stalk cohomology of $\phi_L[-1]\mathbb{Z}_X^\bullet[n]$ gives the reduced cohomology of the **complex link** of X, denoted $\mathbb{L}_{X,\mathbf{0}}$. Using $\mathbb{Q}$ coefficients, we can assume that Y is either smooth or a rational homology manifold, and all the results of [9] follow through without change.

By a classical theorem of Lê, the complex link of X at $\mathbf{0}$ has the homotopy type of a bouquet of n-spheres, and we can detect number of such spheres with the Milnor number of the restriction of a linear form $L_{|X}$ at $\mathbf{0}$, provided that the linear form L is sufficiently generic. For our purposes, the Milnor number will be defined as

$$\mu_{\mathbf{0}}\left(L_{|X}\right) = \operatorname{rank} H^0(\phi_L[-1]\mathbb{Z}_X^\bullet[n])_{\mathbf{0}}.$$

Theorem 1.2 (H., Massey '16)**.** *Suppose that $\mathbf{0}$ is an isolated point in $\operatorname{supp}\phi_L[-1]\mathbb{Q}_X^\bullet[n]$. Then, the Milnor number $\mu_{\mathbf{0}}(L_{|X})$ of L restricted to X at $\mathbf{0}$ is given by:*

(1)

$$\mu_{\mathbf{0}}(L_{|X}) = \Big[\sum_{p\in\pi^{-1}(\mathbf{0})} \mu_p(L\circ\pi)\Big] + (-1)^{n-1}\Big[m(\mathbf{0}) - \sum_{k\geq 2}(k-1)\chi(m^{-1}(k)\cap\mathbb{L}_{X,\mathbf{0}})\Big].$$

(2) In particular, if $\mathbf{0}$ is an isolated point in $\operatorname{supp}\phi_L[-1]\mathbb{Q}_X^\bullet[n]$ and $\pi^{-1}(\mathbf{0})\cap\Sigma(L\circ\pi) = \emptyset$, then

$$\mu_{\mathbf{0}}(L_{|X}) = H^0(\phi_L[-1]\mathbf{N}_X^\bullet)_{\mathbf{0}} = (-1)^{n-1}\Big[m(\mathbf{0}) - \sum_{k\geq 2}(k-1)\chi(m^{-1}(k)\cap\mathbb{L}_{X,\mathbf{0}})\Big].$$

Remark 1.3. When π is additionally a **one-parameter unfolding** with parameter L, $\mu_{\mathbf{0}}(L_{|X})$ coincides with the **image Milnor number** of David Mond [19], when n is a "nice dimension."

As a further application, we quickly recover a classical formula for the Milnor number, as given in Theorem 10.5 of [18]:

Theorem 1.4 (Milnor, '68)**.** *Suppose $(V(f_0),\mathbf{0}) \subseteq (\mathbb{C}^2,\mathbf{0})$ is a plane curve singularity with r irreducible components at the origin, with δ double points appearing in a stable deformation of $V(f_0)$. Then, the Milnor number of f_0 is given by the formula:*

$$\mu_{\mathbf{0}}\left(f_0\right) = 2\delta - r + 1.$$

See Theorem 5.3 of [9] for our proof.

1.2. **Generalizing Milnor's Double Point Formula.** In a recent paper [6], I was able to generalize Milnor's formula (Theorem 1.4) for the Milnor number of a plane curve singularity to deformations of hypersurfaces of arbitrary dimension with smooth normalizations (and, if one is willing to work with $\mathbb{Q}$ coefficients, this generalization holds for hypersurfaces whose normalization is a $\mathbb{Q}$-homology manifold as well), which we have referred to in [9] and [6] as parameterizations of hypersurfaces. Precisely, I give a formula for the **Lê numbers** [15] of the special fiber of this deformation in terms of the Lê numbers of the generic fiber, together with the characteristic polar multiplicities of the comparison complex on the hypersurface.

In this set-up, suppose now that $\mathcal{W}$ is an open neighborhood of the origin in $\mathbb{C}^n$, $\mathbb{D}^\circ$ is an open disk around the origin in $\mathbb{C}$, and $f : (\mathbb{D}^\circ \times \mathcal{U}, \mathbf{0}) \to (\mathbb{C}, 0)$ is a reduced complex analytic function. Thus, f defines a one-parameter analytic family $f_t(\mathbf{z}) := f(t, \mathbf{z})$. Additionally, assume $\pi : (\mathbb{D}^\circ \times \mathcal{W}, \mathbb{D}^\circ \times S) \to (\mathbb{D}^\circ \times \mathcal{U}, \mathbf{0})$ is a finite morphism that is generically one-to-one with $\operatorname{im} \pi = V(f)$, of the form $\pi(t, \mathbf{z}) = (t, \pi_t(t, \mathbf{z}))$; i.e., π is a **one-parameter unfolding of the map π_0, which a generically one-to-one parameterization of** $V(f, t)$. That is, π is a parameterization of the total space of the unfolded hypersurface $V(f)$, or equivalently, π is a simultaneous normalization of the family of hypersurfaces $V(f_t)$, and $S = \pi^{-1}(\mathbf{0})$ is a finite subset of $\mathcal{W}$, an open subset of $\mathbb{C}^{n-1}$.

The central tool used to prove this result, the characteristic polar multiplicities of a perverse sheaf, were first defined and explored in [16]. These multiplicities, defined using nearby and iterated vanishing cycles with respect to a "nice" choice of a tuple of linear forms $\mathbf{z} = (z_0, \cdots, z_s)$, are non-negative integer-valued functions that mimic a characterization of the Lê numbers associated to non-isolated hypersurface singularities (see [15]), and generalize the classical Milnor number of an isolated singularity.

Definition 1.5. *Let $\mathbf{P}^\bullet$ be a perverse sheaf on $V(f)$, with $\dim_{\mathbf{0}} \operatorname{supp} \mathbf{P}^\bullet = s$. Let $\mathbf{z} = (z_0, \cdots, z_s)$ be a tuple of linear forms such that, for all $0 \leq i \leq s$, we have*

$$\dim_{\mathbf{0}} \operatorname{supp} \phi_{z_i - z_i(p)}[-1]\psi_{z_{i-1}-z_{i-1}(p)}[-1] \cdots \psi_{z_0 - z_0(p)}[-1]\mathbf{P}^\bullet \leq 0.$$

Then, the i-dimensional ***characteristic polar multiplicity of*** $\mathbf{P}^\bullet$ *with respect to $\mathbf{z}$ at $p \in V(f)$ is given by the formula*

$$\lambda^i_{\mathbf{P}^\bullet, \mathbf{z}}(p) = \operatorname{rank}_{\mathbb{Z}} H^0(\phi_{z_i - z_i(p)}[-1]\psi_{z_{i-1}-z_{i-1}(p)}[-1] \cdots \psi_{z_0 - z_0(p)}\mathbf{P}^\bullet)_p.$$

Finally, we must specify how we deform these parameterized hypersurfaces. We do not necessarily have a deformation into something as nice as double-points, as in the curve case. We choose the notion of a deformation with isolated polar activity at the origin (or, an **IPA-deformation** [14])–these are deformations which, intuitively, are those where the only "interesting" behavior happens at the origin (see Proposition 2.5 of [6] for a precise definition).

Theorem 1.6 (H. '17)**.** *Suppose that $\pi : (\mathcal{W}, \{0\} \times S) \to (\mathcal{U}, \mathbf{0})$ is a one-parameter unfolding with parameter t, with $\operatorname{im} \pi = V(f)$ for some $f \in \mathcal{O}_{\mathcal{U}, \mathbf{0}}$. Suppose further that $\mathbf{z} = (z_1, \cdots, z_n)$ is chosen such that $\mathbf{z}$ is an IPA-tuple for $f_0 = f_{|_{V(t)}}$ at $\mathbf{0}$. Then, the following relationship holds between the Lê numbers of f_{t_0} with respect to $\mathbf{z}$ at $\mathbf{0}$ and the characteristic polar multiplicities of $\mathbf{N}^\bullet_{V(f_{t_0})} := \mathbf{N}^\bullet_{V(f)_{|_{V(t-t_0)}}}[-1]$: for $0 \leq i \leq n-2$, and for $0 < |t_0| \ll \epsilon \ll 1$,*

$$\lambda^i_{f_0, \mathbf{z}}(\mathbf{0}) + \lambda^i_{\mathbf{N}^\bullet_{V(f_0)}, \mathbf{z}}(\mathbf{0}) = \sum_{p \in B_\epsilon \cap V(t - t_0, z_1, z_2, \cdots, z_i)} \left(\lambda^i_{f_{t_0}, \mathbf{z}}(p) + \lambda^i_{\mathbf{N}^\bullet_{V(f_{t_0})}, \mathbf{z}}(p) \right)$$

One easily checks that, for a parameterized hypersurface $V(f)$ with one-dimensional critical locus, this equality is only non-trivial for $i = 0$, and in that dimension we recover Milnor's original formula.

1.3. **The Comparison Complex as a Mixed Hodge Module.** By shrinking $\mathcal{U}$ if necessary, the perverse sheaf $\mathbb{Q}^\bullet_{V(f)}[n]$ underlies a graded-polarizable mixed Hodge module (Prop 2.19, Prop 2.20, [20]) of weight $\leq n$. Moreover, by Saito's theory of (graded-polarizable) mixed Hodge modules in the local complex analytic context, the perverse cohomology objects of the usual sheaf functors naturally lift to cohomology functors in the context of (graded-polarizable) mixed Hodge modules (but not on their derived category level as in the algebraic context as in Section 4 of [20]). Moverover, by (4.5.9) [20], the quotient morphism $\mathbb{Q}^\bullet_{V(f)}[n] \to \mathbf{I}^\bullet_{V(f)}$ induces an isomorphism

$$\operatorname{Gr}^W_n \mathbb{Q}^\bullet_{V(f)}[n] \xrightarrow{\sim} \mathbf{I}^\bullet_{V(f)};$$

consequently, the short exact sequence (1) identifies the comparison complex $\mathbf{N}^\bullet_{V(f)}$ with $W_{n-1}\mathbb{Q}^\bullet_{V(f)}[n]$. This then endows $\mathbf{N}^\bullet_{V(f)}$ with the structure of a mixed Hodge module of weight $\leq n-1$ with weight filtration $W_k\mathbf{N}^\bullet_{V(f)} = W_k\mathbb{Q}^\bullet_{V(f)}[n]$ for $k \leq n-1$. In a recent paper [8], I have explicitly identified the graded piece $\mathrm{Gr}^W_{n-1}\,\mathbf{N}^\bullet_{V(f)} = \mathrm{Gr}^W_{n-1}\,\mathbb{Q}^\bullet_{V(f)}$ in the case where the normalization of $V(f)$ is a rational homology manifold, and give concrete computations of $W_{n-2}\mathbb{Q}^\bullet_{V(f)}[n]$ in the case where $V(f)$ is a surface in $\mathbb{C}^3$.

Let $i : \Sigma f \hookrightarrow V(f)$. We can then find a smooth, Zariski open dense subset $\mathcal{V} \subseteq \Sigma f$ over which the normalization map restricts to a covering projection $\hat{\pi} : \pi^{-1}(\mathcal{V}) \to \mathcal{V} \subseteq \Sigma f$. Let $l : \mathcal{V} \hookrightarrow \Sigma f$ and $m : \Sigma f \backslash \mathcal{V} \hookrightarrow \Sigma f$ denote the respective open and closed inclusion maps. Let $\hat{m} := i \circ m$, $\hat{l} := i \circ l$. The main result of [8] is the following.

Theorem 1.7 ([8], Theorem 2.6)**.** *Suppose the normalization of $V(f)$ is a rational homology manifold. Then, there is an isomorphism $\mathrm{Gr}^W_{n-1}\, i^*\mathbf{N}^\bullet_{V(f)} \cong \mathbf{I}^\bullet_{\Sigma f}(\hat{l}^*\mathbf{N}^\bullet_{V(f)})$, so that the short exact sequence of perverse sheaves on $V(f)$*

$$0 \to m_*{}^pH^0(m^!i^*\mathbf{N}^\bullet_{V(f)}) \to i^*\mathbf{N}^\bullet_{V(f)} \to \mathbf{I}^\bullet_{\Sigma f}(\hat{l}^*\mathbf{N}^\bullet_{V(f)}) \to 0$$

identifies $W_{n-2}i^*\mathbf{N}^\bullet_{V(f)} \cong m_*{}^pH^0(m^!i^*\mathbf{N}^\bullet_{V(f)})$. Here, $\mathbf{I}^\bullet_{\Sigma f}(\hat{l}^*\mathbf{N}^\bullet_{V(f)})$ denotes the intermediate extension of the perverse sheaf $\hat{l}^*\mathbf{N}^\bullet_{V(f)}$ to all of Σf, and ${}^pH^0(-)$ denotes the 0-th perverse cohomology functor.

Since the map $i : \Sigma f \hookrightarrow V(f)$ is a closed inclusion, it preserves weights. Moreover, the support of $\mathbf{N}^\bullet_{V(f)}$ is contained in the singular locus Σf, and so $i_*i^*\mathbf{N}^\bullet_{V(f)} \cong \mathbf{N}^\bullet_{V(f)}$. Consequently, we have the following.

Corollary 1.8 ([8], Corollary 2.7)**.** *Suppose the normalization of $V(f)$ is a rational homology manifold. Then, there are isomorphisms*

$$\mathrm{Gr}^W_{n-1}\,\mathbb{Q}^\bullet_{V(f)}[n] \cong \mathrm{Gr}^W_{n-1}\,\mathbf{N}^\bullet_{V(f)} \cong i_*\mathbf{I}^\bullet_{\Sigma f}(l^*\mathbf{N}^\bullet_{V(f)}),$$

and

$$W_{n-2}\mathbb{Q}^\bullet_{V(f)}[n] \cong W_{n-2}\mathbf{N}^\bullet_{V(f)} \cong \hat{m}_*{}^pH^0(m^!i^*\mathbf{N}^\bullet_{V(f)}).$$

In the case where $V(f)$ is a surface in $\mathbb{C}^3$, we explicitly compute $W_0\mathbb{Q}^\bullet_{V(f)}[2]$; the vanishing of this perverse sheaf places strong constraints on the topology of the singular set Σf of $V(f)$.

Theorem 1.9 ([8], Theorem 3.1)**.** *If $V(f)$ is a surface in $\mathbb{C}^3$ whose normalization is a rational homology manifold, and $\dim_{\mathbf{0}}\Sigma f = 1$, then*

$$W_0\mathbb{Q}^\bullet_{V(f)}[2] \cong V^\bullet_{\{\mathbf{0}\}}$$

is a perverse sheaf concentrated on a single point, i.e., a finite-dimensional $\mathbb{Q}$-vector space, of dimension

$$\dim_{\mathbb{Q}} V = 1 - |\pi^{-1}(\mathbf{0})| + \sum_C \dim_{\mathbb{Q}} \ker\{\mathrm{id} - h_C\},$$

where $\{C\}$ is the collection of irreducible components of Σf at $\mathbf{0}$, and for each C, h_C is the (internal) monodromy operator on the local system $H^{-1}(\mathbf{N}^\bullet_{V(f)})_{|C\backslash\{\mathbf{0}\}}$. Note that $|\pi^{-1}(\mathbf{0})|$ is, of course, equal to the number of irreducible components of $V(f)$ at $\mathbf{0}$.

Remark 1.10. One immediately notices that $W_0\mathbb{Q}^\bullet_{V(f)}[2]$ bears a striking resemblance to the vanishing cycles $\phi_L[-1]\mathbf{N}^\bullet_{V(f)}$ for L a generic linear form on $\mathbb{C}^3$. In particular, it is easy to see that, for such an L, $W_0\mathbb{Q}^\bullet_{V(f)}[2]$ is a perverse subobject of $\phi_L[-1]\mathbf{N}^\bullet_{V(f)}$, i.e., the vector space V in Theorem 1.9 is a natural subspace of $H^0(\phi_L[-1]\mathbf{N}^\bullet_{V(f)})_{\mathbf{0}}$.

Question 1. Additionally, one notes that the dimension of the vector space V is very similar to the beta invariant (4) below, as $\chi(\mathbf{N}^\bullet_{V(f)})_{\mathbf{0}} = 1 - |\pi^{-1}(\mathbf{0})|$. Does its vanishing have a similar geometric significance to the vanishing of β_f?

It is possible for $V = 0$; this happens, e.g., for the Whitney umbrella $V(y^2 - x^3 - zx^2)$ for which Σf is smooth at the origin, but this is not a sufficient condition. Indeed, the critical locus of $V(xz^2 - y^3)$ is smooth at $\mathbf{0}$, but $V = \mathbb{Q}$. However, we may distinguish these examples by noting that, for generic linear forms L, the normalization map $\pi : Y \to V(f)$ is a **simultaneous normalization** of the the family $\pi_\xi : Y \cap (L \circ \pi)^{-1}(\xi) \to V(f, L - \xi)$ for all $\xi \in \mathbb{C}$ small in the case of the Whitney umbrella, but **not** for the surface $V(xz^2 - y^3)$. Is this true in general? This would make the perverse sheaf $W_0\mathbb{Q}^\bullet_{V(f)}[2]$ very relevant to Conjecture 7 below.

Understanding the proper generality in which to study $\mathbf{N}^\bullet_X$ is my first project for the future.

Question 2. When X is a local complete intersection (LCI), does $\mathbf{N}^\bullet_X$ have an interpretation analogous to $\ker\{\mathrm{id} - \widetilde{T}_f\}$ in the hypersurface case, in terms of the monodromies of the functions defining the LCI?

Question 3. How much of the relationship between $\mathbf{N}^\bullet_X$ and the vanishing cycles persists in the language of mixed Hodge modules? If we let $N = \frac{1}{2\pi i}\log T_u$ be the logarithm of the unipotent part of the monodromy operator $\widetilde{T}_f$ on the unipotent part of the vanishing cycles, then

$$\ker\{\mathrm{id} - \widetilde{T}_f\} \cong \ker N$$

as perverse sheaves on $V(f)$. In this case, $\ker N$ inherits a weight filtration given by the monodromy weight filtration shifted by n on the unipotent vanishing cycles $\phi_{f,1}[-1]\mathbb{Q}^\bullet_{\mathcal{U}}[n+1]$. Is there a similar description for the graded pieces of this weight filtration as in Theorem 1.7 in general, and Theorem 1.9 in the surface case?

Question 4. In the set-up of Theorem 1.6, almost all of the machinery still works for local complete intersections, but it is no longer clear what might replace the "Milnor Fiber", since these only exist in general for isolated complete intersection singularities. In this context, what is the topological significance of the characteristic polar multiplicities of $\mathbf{N}^\bullet_X$, in terms of the defining functions of the LCI?

2. Project 2: The Comparison Complex and Lê's Conjecture

My second project concerns a conjecture by Javier Fernández de Bobadilla which is related to a more well-known related conjecture by Lê. Suppose $f : (\mathbb{C}^{n+1}, \mathbf{0}) \to (\mathbb{C}, 0)$ is a complex analytic function germ with $\dim_{\mathbf{0}} \Sigma f = 1$, and denote by $F_{f,\mathbf{0}}$ the Milnor fiber of f at the origin.

Conjecture 5 (Fernández de Bobadilla [2])**.** *Suppose that the reduced integral cohomology $\widetilde{H}^k(F_{f,\mathbf{0}};\mathbb{Z})$ is non-zero only in degree $(n-1)$, and that*

$$\widetilde{H}^{n-1}(F_{f,\mathbf{0}};\mathbb{Z}) \cong \bigoplus_C \mathbb{Z}^{\overset{\circ}{\mu}_C}$$

where the sum is over all irreducible components C of Σf at $\mathbf{0}$, and $\overset{\circ}{\mu}_C$ denotes the generic transverse Milnor number of f along C. Then, in fact, Σf has a single irreducible component, which is smooth.

Remark 2.1. We note that the hypotheses of this conjecture are equivalent to the isomorphism

$$(\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1])_{|\Sigma f} \cong \bigoplus_C i_{C*}(\mathbb{Z}_C^{\overset{\circ}{\mu}_C})^\bullet[1]$$

of perverse sheaves on Σf, where the right-hand side is a trivial local system, and $i_C : C \hookrightarrow \Sigma f$ denotes the closed inclusion.

Massey and I made some progress on this conjecture [10], proving that it does hold in some special cases; these are the first known positive results toward the conjecture. In particular, we prove:

Theorem 2.2 (H., Massey '16)**.** *Bobadilla's Conjecture holds in the following cases.*

(1) Suppose that $\mathcal{U}$ and $\mathcal{W}$ are open neighborhoods of the origin in $\mathbb{C}^{n+1}$ and $\mathbb{C}^{m+1}$, respectively, and let $g : (\mathcal{U}, \mathbf{0}) \to (\mathbb{C}, 0)$ and $h : (\mathcal{W}, \mathbf{0}) \to (\mathbb{C}, 0)$ be two complex analytic functions. Let $\pi_1 : \mathcal{U} \times \mathcal{W} \to \mathcal{U}$ and $\pi_2 : \mathcal{U} \times \mathcal{W} \to \mathcal{W}$ be the natural projection maps, and set $f = g \boxplus h := g \circ \pi_1 + h \circ \pi_2$. Suppose that g has a one-dimensional critical locus at the origin, and that h has an isolated critical point at the origin.

Then, Bobadilla's Conjecture is true for g if and only if it is true for f.

*(2) Suppose that the **relative polar curve** of f (with respect to a generic linear form z_0) is a hypersurface inside the **relative polar surface** of f. Then, Bobadilla's Conjecture is true for f.*

In particular, the conjecture is true for any non-reduced plane curve singularity.

We approach this conjecture via a new numerical invariant, called the **beta invariant** of the singularity, first explicitly defined and explored in [11], although it appeared indirectly in the literature as early as [23]. Precisely, the beta invariant of a function f with one-dimensional critical locus at $\mathbf{0}$ is

$$\beta_f := \lambda^0_{f,L}(\mathbf{0}) - \lambda^1_{f,L}(\mathbf{0}) + \sum_C \overset{\circ}{\mu}_C = \chi(\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1])_{\mathbf{0}} + \sum_C \overset{\circ}{\mu}_C, \tag{4}$$

where L is any linear form such that $\dim_{\mathbf{0}} \Sigma\left(f_{|_{V(L)}}\right) \leq 0$. The beta invariant is independent of the linear form chosen, and is an invariant of the local, ambient topological-type of the hypersurface $V(f)$ at $\mathbf{0}$, and its vanishing is equivalent to the hypotheses of Bobadilla's Conjecture (Theorem 4.1 and Theorem 5.4 of [11], respectively).

Recently, D. Siersma [22] has expanded on our result for non-reduced plane curves to further understand the $\beta_f = 0$ constraint.

Question 6. If $V(f)$ is a parameterized surface in $\mathbb{C}^3$ (in the sense that it has a smooth normalization) with a one-dimensional critical locus, and L is a linear form for which $\dim_{\mathbf{0}} \operatorname{supp} \phi_L[-1]\mathbb{Z}^\bullet_{V(f)}[2] \leq 0$, we may apply the results of Theorem 1.6 to Bobadilla's Conjecture.

For (reduced) surfaces in $\mathbb{C}^3$ with one-dimensonal critical loci at the origin, what is the connection between the beta invariant of f at $\mathbf{0}$ and the formulas in Theorem 1.6?

This conjecture is related to the following more well-known conjecture of Lê:

Conjecture 7 (Lê, [2])**.** *Suppose* $(V(f), \mathbf{0}) \subseteq (\mathbb{C}^3, \mathbf{0})$ *is a reduced hypersurface with* $\dim_{\mathbf{0}} \Sigma f = 1$*, for which the normalization of* $V(f)$ *is a bijection. Then, in fact,* $V(f)$ *is the total space of an equisingular deformation of plane curve singularities.*

I then intend to investigate the following going forward.

Question 8. We note that the assumption of the normalization of $V(f)$ being a bijection is equivalent to $\mathbf{N}^\bullet_{V(f)} = 0$, and that the conjecture is equivalent to the vanishing $\phi_L[-1]\mathbb{Z}^\bullet_{V(f)}[2] = 0$ for generic linear forms L on $\mathbb{C}^3$.

Via Massey's result that $\mathbf{N}^\bullet_{V(f)} \cong \ker\{\mathrm{id} - \widetilde{T}_f\}$, one can additionally show that the conjecture is equivalent to the vanishing of $\operatorname{im}\{\mathrm{id} - T_{L_{|_{V(f)}}}\}$, where $T_{L_{|_{V(f)}}}$ is the Milnor monodromy action on the nearby cycles $\psi_L[-1]\mathbb{Z}^\bullet_{V(f)}[2]$; that is, the vanishing of the **non-unipotent part of the nearby cycles**. What exactly does this vanishing imply about $\mathbf{N}^\bullet_{V(f)}$ and the topology of $V(f)$?

Consequently, in examining both Bobadilla's Conjecture and Lê's Conjecture, one finds oneself again examining the relationship between the comparison complex and the vanishing cycles.

Broader Impacts

The broader impact of my research (and proposed projects) is centered around two themes. My work is specifically in perverse sheaves, but it has applications to many problems in topology and algebraic geometry in general.

The first is that, although the setting of perverse sheaves and the comparison complex is quite specialized, it appears as "the difference" between two fundamental objects: the constant sheaf (from which one recovers the classical singular cohomology of the space), and intersection cohomology (which recovers the Borel-Moore homology of the space, and when one uses field coefficients, is a semi-simple object in the category of perverse sheaves). In this sense, I regard the comparison complex as a fundamental object as well, with far-reaching applications outside of singularity theory and my own field.

The mixed Hodge structure of the cohomology groups of intersection cohomology and the vanishing cycles have also recently seen many interesting applications toward to the mathematics of string theory [1],[4],[17]. Consequently, the comparison complex underlies a mixed Hodge module with a broad applications to this field.

Second, Lê's conjecture is a decades-old problem of interest to algebraic geometers in general, in the vein of classical equisingularity problems of Mumford and Zariski. Any progress toward this conjecture stands to benefit many different areas of mathematics.

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