# SOME FUN PROBLEMS
# WITH PERVERSE T-STRUCTURES:
# INTERMEDIATE EXTENSIONS

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Let $\mathcal{U}$ be a complex analytic set. Throughout, we use the (somewhat sloppy) convention of writing $\mathbf{A}^\bullet \in D_c^b(\mathcal{U})$, when one should really write $\mathbf{A}^\bullet \in Ob(D_c^b(\mathcal{U}))$. Let $({}^\mu D^{\leq 0}(\mathcal{U}), {}^\mu D^{\geq 0}(\mathcal{U}))$ be the perverse $t$-structure on $D_c^b(\mathcal{U})$ with respect to the middle perversity $\mu$ and associated truncation functors

$$
\begin{aligned}
{}^\mu\tau^{\leq 0} &: D_c^b(\mathcal{U}) \to {}^\mu D^{\leq 0}(\mathcal{U}) \\
{}^\mu\tau^{\geq 0} &: D_c^b(\mathcal{U}) \to {}^\mu D^{\geq 0}(\mathcal{U})
\end{aligned}
$$

When the superscript $\mu$ is omitted, we will mean the so-called "standard" $t$-structure on $D_c^b(\mathcal{U})$, $(D^{\leq 0}(\mathcal{U}), D^{\geq 0}(\mathcal{U}))$, where

$$
\begin{aligned}
Ob(D^{\leq 0}(\mathcal{U})) &= \{\mathbf{A}^\bullet \in D_c^b(\mathcal{U}) | \text{ For all } k > 0, \mathbf{H}^k(\mathbf{A}^\bullet) = 0\} \\
Ob(D^{\geq 0}(\mathcal{U})) &= \{\mathbf{A}^\bullet \in D_c^b(\mathcal{U}) | \text{ For all } k < 0, \mathbf{H}^k(\mathbf{A}^\bullet) = 0\}
\end{aligned}
$$

and $\mathbf{H}^k(-)$ denote the usual cohomology sheaf functors on $D_c^b(\mathcal{U})$. $Perv(\mathcal{U}) := {}^\mu D^{\leq 0}(\mathcal{U}) \cap {}^\mu D^{\geq 0}(\mathcal{U})$, the heart of the perverse $t$-structure on $D_c^b(\mathcal{U})$, is the category of *perverse sheaves* on $\mathcal{U}$. Finally, for integers $a, b$ with $a \leq b$, we set $D^{[a,b]}(\mathcal{U}) := D^{\geq a}(\mathcal{U}) \cap D^{\leq b}(\mathcal{U})$.

To avoid as much confusion as possible, we introduce the following conventions concerning various $t$-structures. Let $X$ and $Y$ be two topological spaces, and let $F : D^b(X) \to D^b(Y)$ be a functor of triangulated categories (i.e., $F$ is additive, commutes with the "shift", and takes distinguished triangles to distinguished triangles). When speaking of the left (resp., right) $t$-exactness of $F$ with respect to the **standard** $t$-structures on $X$ and $Y$, we will simply say $F$ is left (resp., right) exact, with the understanding that that we mean $F(D^{\geq 0}(X)) \subseteq D^{\geq 0}(Y)$ (resp., $F(D^{\leq 0}(X)) \subseteq D^{\leq 0}(Y)$)).

If, in addition, $X$ and $Y$ are complex analytic spaces, when speaking of the left (resp., right) $t$-exactness of $F$ with respect to the **perverse** $t$-structure (with respect to the middle perversity $\mu$) on $X$ and $Y$, we shall say $F$ is left (resp., right) $\mu$-exact, with the understanding that we mean $F({}^\mu D^{\geq 0}(X)) \subseteq {}^\mu D^{\geq 0}(Y)$ (resp., $F({}^\mu D^{\leq 0}(X)) \subseteq {}^\mu D^{\leq 0}(Y)$).

## 1. The Intermediate Extension: Definition and Basic Properties

The following (somewhat general) lemmas will come in handy later on:

**Lemma 1.1.** *Let $\mathcal{D}$ be a triangulated category equipped with $t$-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. Let $X \in \mathcal{D}$, and suppose there is a distinguished triangle*

$$X_0 \to X \to X_1 \xrightarrow{+1} \tag{1}$$

*with $X_0 \in \mathcal{D}^{\leq 0}$ and $X_1 \in \mathcal{D}^{\geq 0}$. Then, $X_0 \cong \tau^{\leq 0} X$ and $X_1 \cong \tau^{\geq 1} X$.*

*Proof.* Recall that the natural morphism $\tau^{\leq 0} X \xrightarrow{\alpha} X$ is the counit of the inclusion-truncation adjunction between the categories $\mathcal{D}^{\leq 0}$ and $\mathcal{D}$, and therefore satisfies the following universal property:

for any morphism $Y \xrightarrow{\beta} X$ with $Y \in \mathcal{D}^{\leq 0}$, there exists a unique morphism $Y \xrightarrow{\gamma} \tau^{\leq 0}X$ such that $\beta = \alpha \circ \gamma$. Equivalently [**REFERENCE**, adjunctions], for $Y \in \mathcal{D}^{\leq 0}$, there is an isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(Y, X) \cong \mathrm{Hom}_{\mathcal{D}}(Y, \tau^{\leq 0}X)$$

Applying the cohomological functor $\mathrm{Hom}_{\mathcal{D}}(Y, -)$ (for $Y \in \mathcal{D}^{\leq 0}$) to the distinguished triangle (1), there exists a long exact sequence

$$\cdots \to \mathrm{Hom}_{\mathcal{D}}(Y, X_1[-1]) \to \mathrm{Hom}_{\mathcal{D}}(Y, X_0) \to \mathrm{Hom}_{\mathcal{D}}(Y, X) \to \mathrm{Hom}_{\mathcal{D}}(Y, X_1) \to \cdots$$

Since $X_1 \in \mathcal{D}^{\geq 1}, \mathrm{Hom}_{\mathcal{D}}(Y, X_1[-1]) = \mathrm{Hom}_{\mathcal{D}}(Y, X_1) = 0$, we get an isomorphism $\mathrm{Hom}_{\mathcal{D}}(Y, X_0) \cong \mathrm{Hom}_{\mathcal{D}}(Y, X)$. That is, the morphism $X_0 \to X$ satisfies the same universal property as $\tau^{\leq 0} \to X$, and therefore there is a unique ismorphism $X_0 \cong \tau^{\leq 0}X$.

The dual claim, that $X_1 \cong \tau^{\geq 1}X$, follows from exactly the same type of argument as the above. That is, by applying the functor $\mathrm{Hom}_{\mathcal{D}}(-, Z)$ (for $Z \in \mathcal{D}^{\geq 1}$) to the distinguished triangle (1), and applying the axioms for a triangulated category.

□

**Theorem 1.1.** *Let $Y$ be a closed complex analytic subset of a complex analytic space $X$. Let $i : Y \hookrightarrow X$ and $j : \mathcal{U} = X - Y \hookrightarrow X$ be the usual inclusion maps. For $\boldsymbol{A}^\bullet \in Perv(\mathcal{U})$, we set*

$$j_{!*}\boldsymbol{A}^\bullet := \mathrm{Im}\left({}^\mu H^0(j_!\boldsymbol{A}^\bullet) \to {}^\mu H^0(Rj_*\boldsymbol{A}^\bullet)\right)$$

*arising from the natural morphism $j_!\boldsymbol{A}^\bullet \to Rj_*\boldsymbol{A}^\bullet$ in $D_c^b(X)$. Then, $j_{!*}\boldsymbol{A}^\bullet$ is the unique (up to quasi-isomorphism) perverse sheaf on $X$ that agrees with $\boldsymbol{A}^\bullet$ on $\mathcal{U}$, and has no non-zero perverse sub- or quotient objects whose support is contained in $Y$.*

Before we give a proof, we wish to prove the following technical lemma:

**Lemma 1.2.** *Suppose $Y$ and $X$ are as in the above claim, and let $\boldsymbol{A}^\bullet \in Perv(X)$. Then, $\boldsymbol{A}^\bullet$ has no non-zero perverse subobjects whose support is contained in $Y$ if and only if ${}^\mu H^0(i^!\boldsymbol{A}^\bullet) = 0$.*

*Dually, $\boldsymbol{A}^\bullet$ has no non-zero perverse quotient objects whose support is contained in $Y$ if and only if ${}^\mu H^0(i^*\boldsymbol{A}^\bullet) = 0$.*

*Proof.* To make our lives easier, we first wish to characterize those perverse sheaves on $X$ whose support is contained in $Y$. Suppose $\mathbf{A}^\bullet \in Perv(X)$ has support contained in $Y$. Then, there is a distinguished triangle

$$(2) \qquad i_*i^!\mathbf{A}^\bullet \to \mathbf{A}^\bullet \to Rj_*j^*\mathbf{A}^\bullet \xrightarrow{+1}$$

on $X$. Since $j^*\mathbf{A}^\bullet = 0$, applying the perverse cohomology functor ${}^\mu H^0$ to the above triangle immediately demonstrates that $i_*{}^\mu H^0(i^!\mathbf{A}^\bullet) \cong \mathbf{A}^\bullet$. That is, for all $\mathbf{A}^\bullet \in Perv(X)$ with support contained in $Y$, there exists a $\mathbf{B}^\bullet \in Perv(Y)$ such that $i_*\mathbf{B}^\bullet \cong \mathbf{A}^\bullet$. And so, without any loss of generality, when speaking of such objects in $Perv(X)$, we may assume they are of the form $i_*\mathbf{B}^\bullet$ for some $\mathbf{B}^\bullet \in Perv(Y)$.

Now, for $\mathbf{A}^\bullet \in Perv(X)$, suppose $\mathbf{A}^\bullet$ has no non-zero perverse subobjects whose support is contained in $Y$. As above, we have a distinguished triangle on $X$:

$$i_*i^!\mathbf{A}^\bullet \to \mathbf{A}^\bullet \to Rj_*j^*\mathbf{A}^\bullet \xrightarrow{+1}.$$

Since $j^*$ is $\mu$-exact, and $Rj_*$ is left $\mu$-exact, $Rj_*j^*\mathbf{A}^\bullet \in {}^\mu D^{\geq 0}(X)$, and therefore ${}^\mu H^k(Rj_*j^*\mathbf{A}^\bullet) = 0$ for all $k < 0$. Consequently, there is a monomorphism

$$0 \to i_*{}^\mu H^0(i^!\mathbf{A}^\bullet) \to \mathbf{A}^\bullet$$

in $Perv(X)$, given by taking the long exact sequence in perverse cohomology of the distinguished triangle (2). In particular, $i_*{}^\mu H^0(i^!\mathbf{A}^\bullet)$ is a perverse subobject of $\mathbf{A}^\bullet$ whose support is contained in $Y$. Hence, if all such subobjects are zero, then ${}^\mu H^0(i^!\mathbf{A}^\bullet) = 0$, since $i_*$ is fully faithful.

Conversely, suppose ${}^\mu H^0(i^!\mathbf{A}^\bullet) = 0$, and let $\mathbf{B}^\bullet \in Perv(Y)$ be arbitrary. Then, we have $\mathbf{B}^\bullet \cong {}^\mu\tau^{\leq 0}\mathbf{B}^\bullet$, and left $\mu$-exactness of $i^!$ yields the isomorphisms

$$\begin{aligned} \mathrm{Hom}_{D_c^b(X)}(i_*\mathbf{B}^\bullet, \mathbf{A}^\bullet) &\cong \mathrm{Hom}_{D_c^b(Y)}(\mathbf{B}^\bullet, i^!\mathbf{A}^\bullet) \\ &\cong \mathrm{Hom}_{D_c^b(Y)}(\mathbf{B}^\bullet, {}^\mu H^0(i^!\mathbf{A}^\bullet)) \end{aligned}$$

In addition, $i_*$ is fully faithful, so there is an isomorphism

$$\mathrm{Hom}_{D_c^b(Y)}(\mathbf{B}^\bullet, {}^\mu H^0(i^!\mathbf{A}^\bullet)) \cong \mathrm{Hom}_{D_c^b(X)}(i_*\mathbf{B}^\bullet, i_*{}^\mu H^0(i^!\mathbf{A}^\bullet)).$$

Hence, $\mathrm{Hom}_{D_c^b(X)}(i_*\mathbf{B}^\bullet, \mathbf{A}^\bullet) \cong \mathrm{Hom}_{D_c^b(X)}(i_*\mathbf{B}^\bullet, i_*{}^\mu H^0(i^!\mathbf{A}^\bullet))$.

Recall that, if $i_*\mathbf{B}^\bullet$ is a perverse subobject of $\mathbf{A}^\bullet$, there is an associated monomorphism $g : i_*\mathbf{B}^\bullet \to \mathbf{A}^\bullet$ (corresponding to an "inclusion" morphism). However, $g \in \mathrm{Hom}_{D_c^b(X)}(i_*\mathbf{B}^\bullet, \mathbf{A}^\bullet) \cong \mathrm{Hom}_{D_c^b(X)}(i_*\mathbf{B}^\bullet, i_*{}^\mu H^0(i^!\mathbf{A}^\bullet)) = 0$, so $g = 0$. Consequently, $i_*\mathbf{B}^\bullet = 0$.

For the dual statement (concerning perverse quotient objects of $\mathbf{A}^\bullet$), the right $\mu$-exactness of $i^*$ yields the following isomorphisms:

$$\begin{aligned} \mathrm{Hom}_{D_c^b(X)}(\mathbf{A}^\bullet, i_*\mathbf{B}^\bullet) &\cong \mathrm{Hom}_{D_c^b(Y)}(i^*\mathbf{A}^\bullet, \mathbf{B}^\bullet) \\ &\cong \mathrm{Hom}_{D_c^b(Y)}({}^\mu H^0(i^*\mathbf{A}^\bullet), \mathbf{B}^\bullet) \\ &\cong \mathrm{Hom}_{D_c^b(X)}(i_*{}^\mu H^0(i^*\mathbf{A}^\bullet), i_*\mathbf{B}^\bullet) \end{aligned}$$

So, as above, if ${}^\mu H^0(i^*\mathbf{A}^\bullet) = 0$, then $i_*\mathbf{B}^\bullet = 0$. The converse immediately follows from the fact that $i_*{}^\mu H^0(i^*\mathbf{A}^\bullet)$ is itself a perverse quotient object of $\mathbf{A}^\bullet$ whose support is contained in $Y$, which we will now show. Recall there is a canonical distinguished triangle on $X$

$$j_!j^*\mathbf{A}^\bullet \to \mathbf{A}^\bullet \to i_*i^*\mathbf{A}^\bullet \xrightarrow{+1} \tag{3}$$

with $j_!j^*\mathbf{A}^\bullet \in {}^\mu D^{\leq 0}(X)$ since $j^*$ is $\mu$-exact and $j_!$ is right $\mu$-exact. Therefore, ${}^\mu H^k(j_!j^*\mathbf{A}^\bullet) = 0$ for all $k > 0$, which yields an epimorphism

$$\mathbf{A}^\bullet \to i_*{}^\mu H^0(i^*\mathbf{A}^\bullet) \to 0$$

in $Perv(X)$, given by taking the long exact sequence in perverse cohomology of the distinguished triangle (3). Hence, $i_*{}^\mu H^0(i^*\mathbf{A}^\bullet)$ is a perverse quotient object of $\mathbf{A}^\bullet$ whose support is contained in $Y$. So, if all such quotient objects are zero, ${}^\mu H^0(i^*\mathbf{A}^\bullet) = 0$, since $i_*$ is fully faithful.

□

**Remark**: It is now evident that the criterion that a given $\mathbf{A}^\bullet \in Perv(X)$ has no non-zero perverse sub- or quotient objects whose support is contained in $Y$ can now be rephrased as the requirement

$${}^\mu H^0(i^!\mathbf{A}^\bullet) = {}^\mu H^0(i^*\mathbf{A}^\bullet) = 0.$$

Moreover, since $i^!$ is left $\mu$-exact and $i^*$ is right $\mu$-exact, the above "vanishing criterion" is immediately seen to be equivalent to requiring

$$i^!\mathbf{A}^\bullet \in {}^\mu D^{\geq 1}(Y)$$

and

$$i^*\mathbf{A}^\bullet \in {}^\mu D^{\leq -1}(Y)$$

We are now ready to proceed with the proof of Theorem 1.1:

*Proof.* (of **Theorem 1.1**)

Let $\mathbf{A}^\bullet \in Perv(\mathcal{U})$ be given, $j_{!*}\mathbf{A}^\bullet = \mathrm{Im}\left({}^\mu H^0(j_!\mathbf{A}^\bullet) \to {}^\mu H^0(Rj_*\mathbf{A}^\bullet)\right)$. We must first show $j_{!*}\mathbf{A}^\bullet$ indeed satisfies the desired properties, i.e., that $j^*j_{!*}\mathbf{A}^\bullet \cong \mathbf{A}^\bullet$ and ${}^\mu H^0(i^!j_{!*}\mathbf{A}^\bullet) = {}^\mu H^0(i^*j_{!*}\mathbf{A}^\bullet) = 0$.

Checking the natural morphism $j^*Rj_*\mathbf{A}^\bullet \to \mathbf{A}^\bullet$ stalkwise, we have $j^*Rj_*\mathbf{A}^\bullet \cong \mathbf{A}^\bullet$. Similarly, $j^*j_!\mathbf{A}^\bullet \cong \mathbf{A}^\bullet$. Since $j^*$ is $\mu$-exact, it commutes with ${}^\mu H^0$, which yields the desired isomorphism $j^*j_{!*}\mathbf{A}^\bullet \cong \mathbf{A}^\bullet$.

Recall that the induced natural morphism ${}^\mu H^0(j_!\mathbf{A}^\bullet) \to {}^\mu H^0(Rj_*\mathbf{A}^\bullet)$ has a canonical factorization into an epimorphism followed by a monomorphism. That is, the maps ${}^\mu H^0(j_!\mathbf{A}^\bullet) \to j_{!*}\mathbf{A}^\bullet$ and $j_{!*}\mathbf{A}^\bullet \to {}^\mu H^0(Rj_*\mathbf{A}^\bullet)$ are an epimorphism and a monomorphism, respectively (since $Perv(X)$ is an abelian category). If $i_*\mathbf{B}^\bullet$ is a perverse subobject of $j_{!*}\mathbf{A}^\bullet$, it follows that $i_*\mathbf{B}^\bullet$ is a perverse subobject of ${}^\mu H^0(Rj_*\mathbf{A}^\bullet)$, by composing the associated monomorphisms. Dually, if $i_*\mathbf{B}^\bullet$ is a perverse quotient object of $j_{!*}\mathbf{A}^\bullet$, then it is a perverse quotient object of ${}^\mu H^0(j_!\mathbf{A}^\bullet)$ as well, by composing the given epimorphisms.

Thus,${}^\mu H^0(i^{*\mu} H^0(j_!\mathbf{A}^\bullet))) = 0$ implies ${}^\mu H^0(i^*j_{!*}\mathbf{A}^\bullet) = 0$, and ${}^\mu H^0(i^{!\mu} H^0(Rj_*\mathbf{A}^\bullet))) = 0$ implies ${}^\mu H^0(i^!j_{!*}\mathbf{A}^\bullet) = 0$, so it suffices to show

$${}^\mu H^0(i^{*\mu} H^0(j_!\mathbf{A}^\bullet))) = {}^\mu H^0(i^{!\mu} H^0(Rj_*\mathbf{A}^\bullet))) = 0$$

To see this, let $\mathbf{B}^\bullet \in Perv(Y)$ be arbitrary, so we obtain isomorphisms

$$\begin{aligned}\mathrm{Hom}_{D^b_c(X)}(i_*\mathbf{B}^\bullet, {}^\mu H^0(Rj_*\mathbf{A}^\bullet)) &\cong \mathrm{Hom}_{D^b_c(X)}(i_*\mathbf{B}^\bullet, Rj_*\mathbf{A}^\bullet)\\ &\cong \mathrm{Hom}_{D^b_c(Y)}(\mathbf{B}^\bullet, i^!Rj_*\mathbf{A}^\bullet) = 0\end{aligned}$$

as $Rj_*$ is left $\mu$-exact, $i_*$ is $\mu$-exact, and $i^!Rj_* = 0$. Therefore, ${}^\mu H^0(i^{!\mu} H^0(Rj_*\mathbf{A}^\bullet)) = 0$. Dually,

$$\begin{aligned}\mathrm{Hom}_{D^b_c(X)}({}^\mu H^0(j_!\mathbf{A}^\bullet), i_*\mathbf{B}^\bullet) &\cong \mathrm{Hom}_{D^b_c(X)}(j_!\mathbf{A}^\bullet, i_*\mathbf{B}^\bullet)\\ &\cong \mathrm{Hom}_{D^b_c(Y)}(i^*j_!\mathbf{A}^\bullet, \mathbf{B}^\bullet) = 0\end{aligned}$$

since $j_!$ is right $\mu$-exact, $i_*$ is $\mu$-exact, and $i^*j_! = 0$. We then similarly have ${}^\mu H^0(i^{*\mu} H^0(j_!\mathbf{A}^\bullet)) = 0$. So $j_{!*}\mathbf{A}^\bullet$ indeed satisfies the required properties.

We must now show uniqueness. Suppose $\mathbf{IC}^\bullet_X \in Perv(X)$ satisfies $j^*\mathbf{IC}^\bullet_X \cong \mathbf{A}^\bullet$ and ${}^\mu H^0(i^*\mathbf{IC}^\bullet_X) = {}^\mu H^0(i^!\mathbf{IC}^\bullet_X) = 0$. Then, from the adjunction triangle

$$i_*i^!\mathbf{IC}^\bullet_X \to \mathbf{IC}^\bullet_X \to Rj_*j^*\mathbf{IC}^\bullet_X \xrightarrow{+1}$$

we obtain the "rotated" distinguished triangle

$$\mathbf{IC}^\bullet_X \to Rj_*\mathbf{A}^\bullet \to i_*i^!\mathbf{IC}^\bullet_X[1] \xrightarrow{+1} \tag{4}$$

where we also use that $j^*\mathbf{IC}^\bullet_X \cong \mathbf{A}^\bullet$. Applying $i^*$ to (4), we obtain the distinguished triangle

$$i^*\mathbf{IC}^\bullet_X \to i^*Rj_*\mathbf{A}^\bullet \to i^!\mathbf{IC}^\bullet_X[1] \xrightarrow{+1} \tag{5}$$

using that $i^*i_*$ is naturally isomorphic to the identity functor on $D^b_c(Y)$. By **REMARK**, $i^*\mathbf{IC}^\bullet_X \in {}^\mu D^{\leq -1}(Y)$ and $i^!\mathbf{IC}^\bullet_X \in {}^\mu D^{\geq 1}(Y)$ (i.e., $i^!\mathbf{IC}^\bullet_X[1] \in {}^\mu D^{\geq 0}(Y)$), so, by applying Lemma 1.1 to the distinguished triangle (5), we have the following isomorphisms

$$i^*\mathbf{IC}^\bullet_X \cong {}^\mu\tau^{\leq -1} i^*Rj_*\mathbf{A}^\bullet$$

and

$$i^!\mathbf{IC}^\bullet_X[1] \cong {}^\mu\tau^{\geq 0} i^*Rj_*\mathbf{A}^\bullet$$

in $D^b_c(Y)$. Consequently, (4) becomes

$$\mathbf{IC}^\bullet_X \to Rj_*\mathbf{A}^\bullet \to i_*{}^\mu\tau^{\geq 0} i^*Rj_*\mathbf{A}^\bullet \xrightarrow{+1} \tag{6}$$

Most importantly, (6) is obtained by only imposing the requirements $j^*\mathbf{IC}_X^\bullet \cong \mathbf{A}^\bullet$ and ${}^\mu H^0(i^!\mathbf{IC}_X^\bullet) = {}^\mu H^0(i^*\mathbf{IC}_X^\bullet) = 0$ on $\mathbf{IC}_X^\bullet$.

Since $Perv(X)$ is a full triangulated subcategory of $D_c^b(X)$, there exists a morphism $\varphi : \mathbf{IC}_X^\bullet \to j_{!*}\mathbf{A}^\bullet$ such that the following diagram commutes:

$$\begin{array}{ccccccc}
\mathbf{IC}_X^\bullet & \longrightarrow & Rj_*\mathbf{A}^\bullet & \longrightarrow & i_*{}^\mu\tau^{\geq 0}i^*Rj_*\mathbf{A}^\bullet & \longrightarrow & \mathbf{IC}_X^\bullet[1] \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle \varphi[1]} \\
j_{!*}\mathbf{A}^\bullet & \longrightarrow & Rj_*\mathbf{A}^\bullet & \longrightarrow & i_*{}^\mu\tau^{\geq 0}i^*Rj_*\mathbf{A}^\bullet & \longrightarrow & j_{!*}\mathbf{A}^\bullet[1]
\end{array}$$

Consequently, $\varphi$ is an isomorphism in $Perv(X)$ (hence, a quasi-isomorphism of complexes) by an application of the 5-lemma. The theorem then follows. □

## 2. Examples and Calculations

In this section, we wish to get our hands on some "easy" examples of intermediate extensions and their stalk cohomologies.

2.1. **Local Systems with Coefficients in $\mathbb{C}$.** Let $i : \{0\} \hookrightarrow \mathbb{C}$ and $j : \mathbb{C}^* \hookrightarrow \mathbb{C}$ be the inclusion of the origin into $\mathbb{C}$ and the inclusion of its open complement, respectively. Let $\mathcal{L}^\bullet$ be the complex one dimensional local system on $\mathbb{C}^*$ corresponding to the representation $\rho : \pi_1(\mathbb{C}^*, 1) \to Aut(\mathbb{C}) \cong \mathbb{C}^*$ which sends the standard generator of $\pi_1(\mathbb{C}^*, 1) \cong \mathbb{Z}$ to $\lambda \in \mathbb{C}^*$. In addition, recall the shifted complex $\mathcal{L}^\bullet[1]$ is a perverse sheaf on $\mathbb{C}^*$.

Our main goal for this section is to calculate the stalk cohomology groups $\mathbf{H}^k(j_{!*}\mathcal{L}^\bullet[1])_0$. Since $j_{!*}\mathcal{L}^\bullet[1] \in Perv(\mathbb{C})$, we need only consider those values of $k$ with $k \in \{-1, 0\}$.

To begin, we will calculate $\mathbf{H}^k(Rj_*\mathcal{L}^\bullet)_0$, and then use this information to determine $\mathbf{H}^k(j_{!*}\mathcal{L}^\bullet[1])_0$ (out of laziness, we drop the shift by 1 for now, but will include it later on when we need to consider the perversity of $j_{!*}\mathcal{L}^\bullet[1]$ on $\mathbb{C}$). By constructibility of $Rj_*\mathcal{L}^\bullet$, there are isomorphisms

$$\begin{aligned}
\mathbf{H}^k(Rj_*\mathcal{L}^\bullet)_0 &\cong \mathbb{H}^k(B_\varepsilon^\circ; Rj_*\mathcal{L}^\bullet) \\
&\cong \mathbb{H}^k(B_\varepsilon^\circ - \{0\}; \mathcal{L}^\bullet) \\
&\cong \mathbb{H}^k(S^1; \mathcal{L}^\bullet)
\end{aligned}$$

We then proceed by applying the Mayer-Vietoris sequence to the usual open cover $\{\mathcal{U}, \mathcal{V}\}$ of $S^1$, where $\mathcal{U}$ is the "top half of the circle, extending a small amount below the $x$-axis", and $\mathcal{V}$ is the "bottom half of the circle, extending a small amount above the $x$-axis." Then, $\mathcal{U}$ and $\mathcal{V}$ are contractible, and $\mathcal{U} \cap \mathcal{V}$ is homotopic to the disjoint union of two points. The long exact sequence in (hyper)cohomology terminates after $\mathbb{H}^1(S^1; \mathcal{L}^\bullet)$, yielding the exact sequence

$$0 \to \mathbb{H}^0(S^1; \mathcal{L}^\bullet) \to \mathbb{H}^0(\mathcal{U}; \mathcal{L}^\bullet) \oplus \mathbb{H}^0(\mathcal{V}; \mathcal{L}^\bullet) \to \mathbb{H}^0(\mathcal{U} \cap \mathcal{V}; \mathcal{L}^\bullet) \to \mathbb{H}^1(S^1; \mathcal{L}^\bullet) \to 0$$

Let $F$ denote the above morphism $\mathbb{H}^0(\mathcal{U}; \mathcal{L}^\bullet) \oplus \mathbb{H}^0(\mathcal{V}; \mathcal{L}^\bullet) \to \mathbb{H}^0(\mathcal{U} \cap \mathcal{V}; \mathcal{L}^\bullet)$; it is difference of the maps induced by the inclusions $\mathcal{U} \cap \mathcal{V} \hookrightarrow \mathcal{U}$ and $\mathcal{U} \cap \mathcal{V} \hookrightarrow \mathcal{V}$. That is, if

$$\begin{gathered}
\mathbb{H}^0(\mathcal{U}; \mathcal{L}^\bullet) \to \mathbb{H}^0(\mathcal{U} \cap \mathcal{V}; \mathcal{L}^\bullet) \text{ via} \\
\alpha \mapsto (\alpha, \alpha)
\end{gathered}$$

is the map induced by $\mathcal{U} \cap \mathcal{V} \hookrightarrow \mathcal{U}$, and

$$\begin{gathered}
\mathbb{H}^0(\mathcal{V}; \mathcal{L}^\bullet) \to \mathbb{H}^0(\mathcal{U} \cap \mathcal{V}; \mathcal{L}^\bullet) \text{ via} \\
\beta \mapsto (\beta, \lambda\beta)
\end{gathered}$$

is the map induced by $\mathcal{U} \cap \mathcal{V} \hookrightarrow \mathcal{V}$, (where the factor of $\lambda$ is due to monodromy by "going around" $S^1$), then $F$ is given by

$$F : \mathbb{H}^0(\mathcal{U}; \mathcal{L}^\bullet) \oplus \mathbb{H}^0(\mathcal{V}; \mathcal{L}^\bullet) \to \mathbb{H}^0(\mathcal{U} \cap \mathcal{V}; \mathcal{L}^\bullet) \text{ via}$$
$$(\alpha, \beta) \mapsto (\alpha - \beta, \alpha - \lambda\beta)$$

In addition, the morphism $F$ (canonically) identifies $\operatorname{Ker} F$ with $\mathbb{H}^0(S^1; \mathcal{L}^\bullet)$ and $\operatorname{Coker} F$ with $\mathbb{H}^1(S^1; \mathcal{L}^\bullet)$, so we therefore wish to further examine the map $F$. Moreover, one clearly has the isomorphisms $\mathbb{H}^0(\mathcal{U}; \mathcal{L}^\bullet) \cong \mathbb{H}^0(\mathcal{V}; \mathcal{L}^\bullet) \cong \mathbb{C}$ and $\mathbb{H}^0(\mathcal{U} \cap \mathcal{V}; \mathcal{L}^\bullet) \cong \mathbb{C}^2$.

Suppose $(\alpha, \beta) \in \operatorname{Ker} F$, so $\alpha = \beta$, and $\alpha = \lambda\alpha$. We then have two cases: if $\lambda = 1$, or if $\lambda \neq 1$. If $\lambda = 1$, we immediately have the equality $\operatorname{Ker} F = \{(\alpha, \alpha) \in \mathbb{C}^2 | \alpha \in \mathbb{C}\}$. If $\lambda \neq 1$, then since $\mathbb{C}$ is an integral domain, the equality $\alpha = \lambda\alpha$ implies $\alpha = 0$. Consequently, $\operatorname{Ker} F = \{\mathbf{0}\}$.

In order to calculate $\operatorname{Coker} F$, recall that the existence of the exact sequence of vector spaces

$$0 \to \operatorname{Ker} F \to \mathbb{C}^2 \xrightarrow{F} \mathbb{C}^2 \to \operatorname{Coker} F \to 0$$

implies that alternating sum of their dimensions is zero; i.e., $\dim \operatorname{Ker} F - \dim \mathbb{C}^2 + \dim \mathbb{C}^2 - \dim \operatorname{Coker} F = 0$, yielding $\dim \operatorname{Ker} F = \dim \operatorname{Coker} F$. Since we are working over the field $\mathbb{C}$, and have already calculated $\operatorname{Ker} F$, it follows that $\operatorname{Coker} F$ and $\operatorname{Ker} F$ are *abstractly* isomorphic as vector spaces over $\mathbb{C}$. Hence, for $k \in \{0, 1\}$, we have

$$\mathbb{H}^k(S^1; \mathcal{L}^\bullet) \cong \begin{cases} \mathbb{C} & \text{if } \lambda = 1 \\ 0 & \text{if } \lambda \neq 1 \end{cases}$$

We now exploit the fact that, in this example, the intermediate extension of $\mathcal{L}^\bullet[1]$ to $\mathbb{C}$ simplifies quite a bit. Precisely, we prove the following

**Proposition 2.1.** *There is a natural isomorphism $j_{!*}\mathcal{L}^\bullet[1] \cong Rj_*\mathcal{L}^\bullet[1]$ in $D^b_c(\mathbb{C})$.*

*Proof.* The method of the proof is simple: since $j^*Rj_*\mathcal{L}^\bullet[1] \cong \mathcal{L}^\bullet[1]$, if we show

$$i^!Rj_*\mathcal{L}^\bullet[1] \in {}^\mu D^{\geq 1}(\{0\})$$
$$\text{and}$$
$$i^*Rj_*\mathcal{L}^\bullet[1] \in {}^\mu D^{\leq -1}(\{0\})$$

uniqueness of the intermediate extension of $\mathcal{L}^\bullet[1]$ gives the desired isomorphism.

First, note that $i^!Rj_*$ is naturally isomorphic to the "zero" functor, so one trivially has $i^!Rj_*\mathcal{L}^\bullet[1] = 0 \in {}^\mu D^{\geq 1}(\{0\})$. Consider then the complex $i^*Rj_*\mathcal{L}^\bullet[1]$ on $\{0\}$, and recall that, in the proof of uniqueness of $j_{!*}\mathcal{L}^\bullet[1]$, we obtained the isomorphism

$$i^*j_{!*}\mathcal{L}^\bullet[1] \cong {}^\mu\tau^{\leq -1} i^*Rj_*\mathcal{L}^\bullet[1]$$

Since ${}^\mu H^0(i^*j_{!*}\mathcal{L}^\bullet[1]) = 0$, and

$${}^\mu H^0({}^\mu\tau^{\leq -1} i^*Rj_*\mathcal{L}^\bullet[1]) \cong {}^\mu\tau^{\geq 0}({}^\mu\tau^{\leq -1} i^*Rj_*\mathcal{L}^\bullet[1]) = 0,$$

we have the natural isomorphism

$${}^\mu\tau^{\leq -1} i^*Rj_*\mathcal{L}^\bullet[1] \cong i^*Rj_*\mathcal{L}^\bullet[1]$$

implying $i^*Rj_*\mathcal{L}^\bullet[1] \in {}^\mu D^{\leq -1}(\{0\})$ (cf: [KS, Chapter 10]), as desired. Thus, by uniqueness of $j_{!*}\mathcal{L}^\bullet[1]$ on $D^b_c(\mathbb{C})$, $j_{!*}\mathcal{L}^\bullet[1] \cong Rj_*\mathcal{L}^\bullet[1]$, and we are done. □

Hence, $\mathbf{H}^k(j_{!*}\mathcal{L}^\bullet[1])_0 \cong \mathbf{H}^k(Rj_*\mathcal{L}^\bullet[1])_0 \cong \mathbb{H}^{k+1}(S^1; \mathcal{L}^\bullet)$, which we have already calculated above.

2.2. **Local Systems with Coefficients in $\mathbb{Z}$.** As in the previous example, let $i : \{0\} \hookrightarrow \mathbb{C}$ and $j : \mathbb{C}^* \hookrightarrow \mathbb{C}$ be the inclusion maps. Now, let $\mathcal{F}^\bullet$ be the rank one local system with coefficients in $\mathbb{Z}$ on $\mathbb{C}^*$, corresponding to the representation $\rho : \pi_1(\mathbb{C}^*, 1) \to Aut(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ sending the standard generator of $\pi_1(\mathbb{C}^*, 1) \cong \mathbb{Z}$ to either 1 or $-1$ (NB: $\mathbb{Z}/2\mathbb{Z} \cong \{1, -1\}$ with its obvious multiplicative structure). Note that $\mathcal{F}^\bullet[1]$ is a perverse sheaf on $\mathbb{C}^*$.

We wish to compute the stalk cohomology groups $\mathbf{H}^k(j_{!*}\mathcal{F}^\bullet[1])_0$, where $k \in \{-1, 0\}$. We follow the argument in §2.1 *mutatis mutandis* to obtain the exact sequence of $\mathbb{Z}$-modules

$$0 \to \mathbb{H}^0(S^1; \mathcal{F}^\bullet) \to \mathbb{H}^0(\mathcal{U}; \mathcal{F}^\bullet) \oplus \mathbb{H}^0(\mathcal{V}; \mathcal{F}^\bullet) \xrightarrow{G} \mathbb{H}^0(\mathcal{U} \cap \mathcal{V}; \mathcal{F}^\bullet) \to \mathbb{H}^1(S^1; \mathcal{F}^\bullet) \to 0 \tag{7}$$

Which, in this example, is isomorphic to

$$0 \to \mathbb{H}^0(\mathcal{U}; \mathcal{F}^\bullet) \to \mathbb{Z}^2 \xrightarrow{G} \mathbb{Z}^2 \to \mathbb{H}^1(S^1; \mathcal{F}^\bullet) \to 0$$

$$(\alpha, \beta) \stackrel{G}{\mapsto} (\alpha - \beta, \alpha - \epsilon\beta)$$

where $\epsilon = \pm 1$. Furthermore, the morphism $G$ identifies $\mathbb{H}^0(S^1; \mathcal{F}^\bullet)$ with $\operatorname{Ker} G$, and $\mathbb{H}^1(S^1; \mathcal{F}^\bullet)$ with $\operatorname{Coker} G$, and so, to calculate $\mathbf{H}^k(Rj_*\mathcal{F}^\bullet)_0$, it suffices to determine $\operatorname{Ker} G$ and $\operatorname{Coker} G$.

Suppose $(\alpha, \beta) \in \operatorname{Ker} G$, so that $\alpha = \beta$, and $\alpha = \epsilon\alpha$. This leaves us with two cases: if $\epsilon = 1$, or if $\epsilon = -1$. If $\epsilon = 1$, one immediately has $\operatorname{Ker} G = \{(\alpha, \alpha) \in \mathbb{Z}^2 | \alpha \in \mathbb{Z}\} \cong \mathbb{Z}$. Now, suppose $\epsilon = -1$, and let $(\alpha, \beta) \in \operatorname{Ker} G$. Then, $\alpha = \beta$, and $\alpha = -\alpha$. But $\mathbb{Z}$ is an integral domain, so we must have $\alpha = 0$, and therefore must have $\operatorname{Ker} G = \{\mathbf{0}\}$.

We then calculate $\operatorname{Coker} G$. The previous trick, that the the alternating sum of the ranks of the $\mathbb{Z}$-modules in the exact sequence (7) equals zero, again applies, yielding $\operatorname{rank}_\mathbb{Z}(\operatorname{Ker} G) = \operatorname{rank}_\mathbb{Z}(\operatorname{Coker} G)$. Unfortunately, this time we get significantly less information about $\operatorname{Coker} G$; since we are no longer dealing with vector spaces, there might be non-zero torsion elements in $\operatorname{Coker} G$ that are not present in $\operatorname{Ker} G$.

There are then two cases to consider, depending on the sign of $\epsilon$. Suppose first that $\epsilon = 1$. Then, $G(\alpha, \beta) = (\alpha - \beta, \alpha - \beta)$. $G$ then has the matrix representation $G = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ (in the standard basis on $\mathbb{Z}^2$), which, after some linear algebra, reduces to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Therefore, $\operatorname{Coker} G \cong \mathbb{Z}$.

Consider now the case where $\epsilon = -1$, so that $G(\alpha, \beta) = (\alpha - \beta, \alpha + \beta)$. Representing $G$ as the $2 \times 2$ integer matrix $G = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, a little linear algebra shows $G$ is equivalent to the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. In this form, the isomorphism $\operatorname{Coker} G \cong (\mathbb{Z}/\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ is clear.

Therefore,

$$\mathbb{H}^0(S^1; \mathcal{F}^\bullet) \cong \begin{cases} \mathbb{Z} & \text{if } \epsilon = 1 \\ 0 & \text{if } \epsilon = -1 \end{cases}$$

and

$$\mathbb{H}^1(S^1; \mathcal{F}^\bullet) \cong \begin{cases} \mathbb{Z} & \text{if } \epsilon = 1 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } \epsilon = -1 \end{cases}$$

Finally, we remark that scrutiny of the proof of Proposition 2.1 reveals that the result holds over $\mathbb{Z}$ as well; hence, $Rj_*\mathcal{F}^\bullet[1] \cong j_{!*}\mathcal{F}^\bullet[1]$, yielding

$$\mathbf{H}^k(j_{!*}\mathcal{F}^\bullet[1])_0 \cong \mathbf{H}^k(Rj_*\mathcal{F}^\bullet[1])_0 \cong \mathbb{H}^{k+1}(S^1; \mathcal{F}^\bullet)$$

and we are done, by our above calculations.