The Fundamental Short Exact Sequence

Suppose $X$ is a purely $d$-dimensional local complete intersection inside some neighborhood of the origin in $\mathbb{C}^N$. Then, the shifted constant sheaf $\mathbb{Z}[d]$ is perverse, and there is a canonical monomorphism $\mathbb{Z}[d] \to \mathbb{P}$, where $\mathbb{P}$ is the intersection cohomology complex on $X$ with constant $\mathbb{Z}$ coefficients. Since $\mathbb{P}$ is also the intermediate extension of the constant sheaf on $X_{d+1}$, it has no trivial sub-perverses or quotient-verse sheaves with support contained in $X_{d+1}$. Therefore, since our morphism induces an isomorphism when restricted to $X_{d+1}$, its cokernel must be zero, i.e., the morphism $\mathbb{P} \to \mathbb{Q}$ is a surjection.

We let $\mathbb{N}_X$ be the kernel of the morphism $\mathbb{c}$, so that there is a short exact sequence of perverse sheaves

$$0 \to \mathbb{N}_X \to \mathbb{Z}[d] \to \mathbb{P} \to 0 \tag{1}$$

on $X$. As $\mathbb{Z}[d]$ and $\mathbb{P}$ are, essentially, the two fundamental perverse sheaves on the LCI $X$, we refer to this as the fundamental short exact sequence of LCI. This short exact sequence, and the perverse sheaf $\mathbb{N}_X$ in particular, have recently been examined in several papers by the author and D. Massey [3], [1], [6], [2]. In particular, we explore several interesting relationships with the vanishing cycles functor and its monodromy, as well as information about the normalization of the LCI that is naturally encoded by the comparison complex.

The Enhanced Support and Cosupport Conditions for the Intermediate Extension

Let $X$ be a purely $(d+1)$-dimensional LCI, $f : X \to C$ a complex analytic function not vanishing on any irreducible component of $X$. Let $Y = V(f)$ be the resulting $d$-dimensional LCI, $j : X \to Y$, and set

$$Y = V(f) \cap X, \quad \mathbb{P} = V(f) \cap \mathbb{Q}, \quad \mathbb{P}_1 = \mathbb{P} / \mathbb{Q} \tag{2}$$

Let $m : Y \to X$, and $m = j \circ m$. Let $(X/Y) \to X$. Then, we can axiomatically define the intermediate extension $\mathbb{P}_1$ of the constant sheaf $\mathbb{Z}[X/d] = \mathbb{Z}[X/\mathbb{Q}]$ across $X$ as the unique perverse sheaf (up to isomorphism) that satisfies

$$\bullet \mathbb{P}_1 \cong \mathbb{Z}[X/d], \quad \text{where } i : X/X \to X;$$

$$\bullet H^i(Y) \mathbb{P}_1 = 0, \quad i \neq 0; \quad \mathbb{P}_1$$

has no quotient-verse sheaves with support in $X$;

$$\bullet H^0(Y) \mathbb{P}_1 = 0, \quad \mathbb{P}_1$$

has no sub-perverses with support in $X$.

Then, say $\mathbb{P}_1$ satisfies the enhanced support condition (along $Y$) if

$$\mu(f, U) \mathbb{P}_1 < 0, \quad \mu(f, V) \mathbb{P}_1 = 0$$

where $\mu(\cdot, \cdot)$ denotes the perverse cohomology functor.

These conditions hold, for example, when:

$$X = U$$

is an open neighborhood of $C^{d+1}$, $X = V(f)$ is a reduced hypersurface, and $Y = \Sigma_f$ is arbitrary.

$$\bar{X} = V(f)$$

is a $Z$-homology hypersurface with an isolated singularity at $0$, $X = V(f,g)$, and

$$Y = V(f) \cap Y_0 \cap V(g) \cap \Sigma_f = 0$$

Theorem 3 (Massey, 2018). If the enhanced support and cosupport conditions hold for $\mathbb{P}_1$ along $Y$, then there is a short exact sequence of perverse sheaves on $X$

$$0 \to \ker (d - f) \mathbb{P}_1 \to \mathbb{P}_1 \to \mathbb{P}_1/d \to 0$$

where $\mathbb{P}_1(d)$ is the intermediate extension of the perverse sheaf $\mathbb{P}_1$ to all of $X$.

Theorem 4 (Massey, 2018). Let $V(f, g) \to V(f)$ and $V(f, g) \to V(f)$ be the natural inclusion maps. If the enhanced support and cosupport conditions hold for $\mathbb{P}_1(f)$ and $\mathbb{P}_1(g)$ along $Y$, then there is an exact sequence

$$0 \to \mathbb{P}_1(f) \to \mathbb{P}_1(g) \to 0$$

where $\mathbb{P}_1(f)$ is the pushout of the natural morphisms

$$\mathbb{P}_1(f) \to \mathbb{P}_1(g) \to \mathbb{P}_1(f, g) \to 0$$

Future Directions

Question 1: Is there a result analogous to $\mathbb{N}_X \cong \ker (I_d - f)$ in the general case of an LCI? How is the comparison complex related (if at all) to the monodromies of the functions defining an LCI?

Question 2: The enhanced support and cosupport conditions on $\mathbb{P}_1$ along $Y$ are quite restrictive, is there a more natural and/or axiomatic way to describe the phenomena in Theorems 3 and 4?

Question 3: When $X$ is a reduced complex algebraic variety of pure dimension $n$, Morihiko Saito [7] has recently shown that

$$W(f, X) \mathbb{P} \cong \mathcal{L} \{ H^i(Y, \mathbb{Q}) \to \mathbb{E}^{-n+1}(X, \mathbb{Q}) \}$$

where $W(f, X, \mathbb{Q})$ denotes the weight part of the cohomology $H^i(Y, \mathbb{Q})$, considered as a mixed Hodge structure, and $\mathbb{Q}$ is the normalization of $X$. How much of the relationship between $\mathbb{N}_X$ and the vanishing cycles (and their monodromy actions) persists in this general setting of arbitrary reduced complex algebraic varieties? What is the extent of the link with Mixed Hodge Modules?

References


The Comparison Complex on a Local Complete Intersection

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Abstract

We examine a perverse sheaf called the comparison complex that is naturally associated to any local complete intersection, first defined and explored in several papers by the author and David Massey [10], [11], [6], [2]. In particular, we explore several interesting relationships with the vanishing cycles functor and its monodromy, as well as information about the normalization of the LCI that is naturally encoded by the comparison complex.