The Comparison Complex on a Local Complete Intersection

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Abstract

We examine a perverse sheaf called the comparison complex that is naturally associated to any local complete intersection, first defined and explored in several papers by the author and David Massey [3], [1], [6], [2]. In particular, we explore several interesting relationships with the vanishing cycles functor and its monodromy, as well as information about the normalization of the LCI that is naturally encoded by the comparison complex.

The Fundamental Short Exact Sequence

Suppose X is a purely d-dimensional local complete intersection inside some open neighborhood of the origin in $\mathbb{C}^N$. Then, the shifted constant sheaf $\mathbb{Z}^\bullet_X[d]$ is perverse, and there is a canonical morphism $\mathbb{Z}^\bullet_X[d] \xrightarrow{c} \mathbf{I}^\bullet_X$, where $\mathbf{I}^\bullet_X$ is the intersection cohomology complex on X with constant $\mathbb{Z}$ coefficients. Since $\mathbf{I}^\bullet_X$ is also the intermediate extension of the constant sheaf on X_{reg}, it has no non-trivial sub-perverse sheaves or quotient-perverse sheaves with support contained in ΣX. Therefore, since our morphism induces an isomorphism when restricted to X_{reg}, its cokernel must be zero, i.e., the morphism c is a surjection.

We let $\mathbf{N}^\bullet_X$ be the kernel of the morphism c, so that there is a short exact sequence of perverse sheaves

$$0 \to \mathbf{N}^\bullet_X \to \mathbb{Z}^\bullet_X[d] \to \mathbf{I}^\bullet_X \to 0 \tag{1}$$

on X. As $\mathbb{Z}^\bullet_X[d]$ and $\mathbf{I}^\bullet_X$ are, essentially, the two fundamental perverse sheaves on the LCI X, we refer to 1 as the **fundamental short exact sequence of the LCI**. This short exact sequence, and the perverse sheaf $\mathbf{N}^\bullet_X$ in particular, have examined recently in several papers by the author and D. Massey in the case where the normalization of X is smooth [3] and [1], and where the normalization is a rational homology manifold [2]. In these papers, we refer to $\mathbf{N}^\bullet_X$ as the **multiple-point complex** of the normalization, as it naturally encodes the data about the image multiple-points of the normalization.

Relationship with the Vanishing Cycles [H.,2017]

In the case where $X = V(f)$ is a hypersurface in some open neighborhood $\mathcal{U}$ of the origin in $\mathbb{C}^{n+1}$, we prove in [1] that a strong relationship holds between the **characteristic polar multiplicities** of $\mathbf{N}^\bullet_X$ and the Lê numbers of the function f. This same result holds for hypersurface normalizations that are $\mathbb{Q}$-homology manifolds. More precisely, the exact same proof yields:

Theorem 1 (H.,2017)**.** *[1] Suppose that $\widetilde{X}$ is a $\mathbb{Q}$-homology manifold, and $\pi : (\widetilde{X} \times \mathbb{C}, \{0\} \times S) \to (\mathcal{U}, \mathbf{0})$ is a one-parameter unfolding with parameter t, with $\operatorname{im}\pi = X = V(f)$ for some $f \in \mathcal{O}_{\mathcal{U},\mathbf{0}}$. Suppose further that $\boldsymbol{z} = (z_1, \cdots, z_n)$ is chosen such that $\boldsymbol{z}$ is an IPA-tuple for $f_0 = f_{|_{V(t)}}$ at $\mathbf{0}$. Then, if $\boldsymbol{N}^\bullet_{X_{t_0}} = \boldsymbol{N}^\bullet_{X|_{V(t-t_0)}}[-1]$, the following formulas hold for the Lê numbers of f_0 with respect to $\boldsymbol{z}$ at $\mathbf{0}$: for $0 < |t_0| \ll \epsilon \ll 1$, for $0 \leq i \leq n-2$,*

$$\lambda^i_{f_0,\boldsymbol{z}}(\mathbf{0}) + \lambda^i_{\boldsymbol{N}^\bullet_{X_0},\boldsymbol{z}}(\mathbf{0}) = \sum_{p \in B_\epsilon \cap V(t-t_0, z_1, z_2, \cdots, z_i)} \left(\lambda^i_{f_{t_0},\boldsymbol{z}}(p) + \lambda^i_{\boldsymbol{N}^\bullet_{X_{t_0}},\boldsymbol{z}}(p) \right)$$

Since the Lê numbers of f are the same as the characteristic polar multiplicities of the vanishing cycles $\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]$, **this result suggests a conservation of number property with the characteristic polar multiplicities of the comparison complex and the vanishing cycles.**

For a precise definition of characteristic polar multiplicities, see [4]; for deformations with isolated polar activity (IPA-deformations and IPA-tuples), see [5].

Relationship with the Monodromy on Vanishing Cycles [Massey, '18]

If one examines the short exact sequence (1) in the case where X is a hypersurface $V(f)$, Massey has recently shown in [6] that there is an isomorphism

$$\mathbf{N}^\bullet_X \cong \ker\{\mathrm{Id} - \widetilde{T}_f\},$$

where $\widetilde{T}_f$ is the monodromy action on the vanishing cycles $\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[d+1]$, and the kernel takes place in the category of perverse sheaves on $V(f)$. In this context, Massey refers to $\mathbf{N}^\bullet_X$ as the **comparison complex** on $V(f)$.

Relationship with $\mathbb{Q}$-Homology Manifolds [H.,2018]

Looking at the short exact sequence (1), one notices immediately that $\mathbb{Q}^\bullet_X[n] \cong \mathbf{I}^\bullet_X$ if and only if $\mathbf{N}^\bullet_X = 0$; that is, the LCI X is a rational homology manifold (or, a **$\mathbb{Q}$-homology manifold**) precisely when the complex $\mathbf{N}^\bullet_X$ vanishes. It is then natural to ask that, given the normalization $\pi : Y \to X$ and the resulting fundamental short exact sequence

$$0 \to \mathbf{N}^\bullet_X \to \mathbb{Q}^\bullet_X[d] \to \pi_*\mathbf{I}^\bullet_Y \to 0,$$

is there a similar result relating $\mathbf{N}^\bullet_X$ to whether or not Y is a $\mathbb{Q}$-homology manifold?

Theorem 2 (H.,2018)**.** *[2] Y is a $\mathbb{Q}$-homology manifold if and only if $\boldsymbol{N}^\bullet_X$ has stalk cohomology concentrated in degree $-d+1$; i.e., for all $p \in X$, $H^k(\boldsymbol{N}^\bullet_X)_p$ is non-zero only possibly when $k = -d+1$.*

The Enhanced Support and Cosupport Conditions for the Intermediate Extension

Let $\widetilde{X}$ be a purely $(d+1)$-dimensional LCI, $f : \widetilde{X} \to \mathbb{C}$ a complex analytic function not vanishing on any irreducible component of $\widetilde{X}$. Let $X = V(f)$ be the resulting d-dimensional LCI, $j : X \hookrightarrow \widetilde{X}$, and set

$$\begin{aligned} Y &:= V(f) \cap \Sigma\widetilde{X} \cup \operatorname{supp}\phi_f[-1]\mathbf{I}^\bullet_{\widetilde{X}} \\ &= V(f) \cap \Sigma\widetilde{X} \cup \operatorname{supp}\phi_f[-1]\mathbb{Z}^\bullet_{\widetilde{X}}[d+1]. \end{aligned}$$

Let $m : Y \hookrightarrow X$, and $\hat{m} = j \circ m$. Let $l : X \backslash Y \hookrightarrow X$. Then, we can axiomatically define the intermediate extension $\mathbf{I}^\bullet_{\widetilde{X}}$ of the constant sheaf $\mathbb{Z}_{\widetilde{X}\backslash X}[d+1]$ across X as the unique perverse sheaf (up to isomorphism) that satisfies

- $i^*\mathbf{I}^\bullet_{\widetilde{X}} \cong \mathbb{Z}^\bullet_{\widetilde{X}\backslash X}[d+1]$, where $i : \widetilde{X}\backslash X \hookrightarrow X$;
- ${}^\mu H^0(j^*\mathbf{I}^\bullet_{\widetilde{X}}) = 0$, i.e., $\mathbf{I}^\bullet_{\widetilde{X}}$ has no quotient-perverse sheaves with support in X;
- ${}^\mu H^0(j^!\mathbf{I}^\bullet_{\widetilde{X}}) = 0$, i.e., $\mathbf{I}^\bullet_{\widetilde{X}}$ has no sub-perverse sheaves with support in X.

Then, say $\mathbf{I}^\bullet_{\widetilde{X}}$ satisfies the **enhanced support condition** (along Y) if

$${}^\mu H^0(\hat{m}^*[-1]\mathbf{I}^\bullet_{\widetilde{X}}) = 0.$$

$\mathbf{I}^\bullet_{\widetilde{X}}$ satisfies the **enhanced cosupport condition** (along Y) if

$${}^\mu H^0(\hat{m}^![1]\mathbf{I}^\bullet_{\widetilde{X}}) = 0,$$

where ${}^\mu H^0(-)$ denotes the perverse cohomology functor.

These conditions hold, for example, when:

- $\widetilde{X} = \mathcal{U}$ is an open neighborhood of $\mathbb{C}^{n+1}$, $X = V(f)$ is a reduced hypersurface, and $Y = \Sigma f$ is arbitrary.
- $\widetilde{X} = V(g)$ is a $\mathbb{Z}$-homology hypersurface with an isolated singularity at $\mathbf{0}$, $X = V(f, g)$, and

$$Y = V(f) \cap \Sigma g \cup V(g) \cap \Sigma f = \{\mathbf{0}\}$$

Theorem 3 (H.,Massey 2018)**.** *If the enhanced support and cosupport conditions hold for $\boldsymbol{I}^\bullet_{\widetilde{X}}$ along Y, then there is a short exact sequence of perverse sheaves on X*

$$0 \to \ker\{\mathrm{Id} - \widetilde{T}_f\}(\boldsymbol{I}^\bullet_{\widetilde{X}}) \to j^*[-1]\boldsymbol{I}^\bullet_{\widetilde{X}} \to \boldsymbol{I}^\bullet_X(l^*\psi_f[-1]\boldsymbol{I}^\bullet_{\widetilde{X}}) \to 0,$$

where $\boldsymbol{I}^\bullet_X(l^*\psi_f[-1]\boldsymbol{I}^\bullet_{\widetilde{X}})$ denotes the intermediate extension of the perverse sheaf $l^*\psi_f[-1]\boldsymbol{I}^\bullet_{\widetilde{X}}$ to all of X.

Theorem 4 (H., Massey 2018)**.** *Let $V(f,g) \overset{j_f}{\hookrightarrow} V(g)$ and $V(f,g) \overset{j_g}{\hookrightarrow} V(f)$ be the natural inclusion maps. If the enhanced support and cosupport conditions hold for $\boldsymbol{I}^\bullet_{V(f)}$ and $\boldsymbol{I}^\bullet_{V(g)}$ along $Y \subseteq V(f,g)$, then there is an exact sequence*

$$\mathbb{Z}^\bullet_{V(f,g)}[d] \rightrightarrows j_f^*[-1]\boldsymbol{I}^\bullet_{V(g)} \oplus j_g^*[-1]\boldsymbol{I}^\bullet_{V(f)} \to \boldsymbol{I}^\bullet_{V(f,g)} \to 0,$$

i.e., $\boldsymbol{I}^\bullet_{V(f,g)}$ is the pushout of the natural morphisms

$$j_f^*[-1]\boldsymbol{I}^\bullet_{V(g)} \leftarrow \mathbb{Z}^\bullet_{V(f,g)}[d] \to j_g^*[-1]\boldsymbol{I}^\bullet_{V(f)}.$$

Future Directions

Question 1: Is there a result analogous to $\mathbf{N}^\bullet_X \cong \mathrm{Ker}\{\mathrm{Id} - \widetilde{T}_f\}$ in the general case of an LCI? **How is the comparison complex related (if at all) to the monodromies of the functions defining an LCI?**

Question 2: The enhanced support and cosupport conditions on $\mathbf{I}^\bullet_{\widetilde{X}}$ along Y are quite restrictive; is there a more natural and/or axiomatic way to describe the phenomena in Theorems 3 and 4?

Question 3: When X is a reduced complex algebraic variety of pure dimension n, Morihiko Saito [7] has recently shown that

$$W_0H^1(X;\mathbb{Q}) \cong \mathrm{Coker}\{H^0(Y;\mathbb{Q}) \to \mathbb{H}^{-n+1}(X;\mathbb{Q})\},$$

where $W_0H^1(X;\mathbb{Q})$ denotes the weight zero part of the cohomology $H^1(X;\mathbb{Q})$, considered as a mixed Hodge Structure, and Y is the normalization of X. **How much of the relationship between $\mathbf{N}^\bullet_X$ and the Vanishing Cycles (and their monodromy actions) persists in this general setting of arbitrary reduced complex algebraic varieties? What is the extent of the link with Mixed Hodge Modules?**

References

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