Perverse Sheaves, Finite Maps, and Numerical Invariants

AMS Special Session on Singularities
Northeastern University

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- We will define a new perverse sheaf, the **multiple-point complex** $\mathbf{N}^\bullet_X$, naturally associated to any "parameterized" LCI [H., Massey 2017]

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- We examine how these invariants "deform" in a one-parameter family (via **one-parameter unfoldings**, or **IPA-deformations**). We compare these deformation formulas with Milnor's classical formula for the Milnor number in terms of double-points.

The Set-Up

- Let $(X, \mathbf{0})$ be the germ of an n-dimensional LCI in some $(\mathbb{C}^N, \mathbf{0})$, and (after picking a suitable representative of X) let $\pi : Y \to X$ be a surjective, finite, and generically one-to-one morphism (e.g. π is a parameterization, or the normalization of X).

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- Then, there is a natural surjection of perverse sheaves $\mathbb{Z}_X^\bullet[n] \to \mathbf{I}_X^\bullet \to 0$ on X, where $\mathbf{I}_X^\bullet$ is the complex of intersection cohomology on X with constant $\mathbb{Z}$-local system. Since $Perv(X)$ is an Abelian category, this morphism has a kernel, $\mathbf{N}_X^\bullet$.

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- Consequently, there is a short exact sequence of perverse sheaves on X:

$$0 \to \mathbf{N}_X^\bullet \to \mathbb{Z}_X^\bullet[n] \to \mathbf{I}_X^\bullet \to 0.$$

Fundamental Short Exact Sequence of π

- Since π is a finite map (really, we just need a **small map** the sense of Goresky and Macpherson), π pushes forward intersection cohomology on Y to intersection cohomology on X, i.e., $\mathbf{I}^\bullet_X \cong \pi_* \mathbf{I}^\bullet_Y$;

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    This means that π **is a one-parameter unfolding of π_0**.
  - Y is the normalization of a LCI X. When Y is additionally a **rational homology manifold**, we call π a $\mathbb{Q}$-**parameterization** of X.

$\mathbf{N}_X^\bullet$ in General

- In general, from the short exact sequence

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  - In degree $-(n-1)$, the stalk cohomology is very easy to describe: for $p \in X$,

$$H^{-(n-1)}(\mathbf{N}_X^\bullet)_p \cong \mathbb{Z}^{m(p)}.$$

where $m(p) := |\pi^{-1}(p)| - 1$, as before.

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    where $m(p) := |\pi^{-1}(p)| - 1$, as before.
- In this case, we call $\mathbf{N}_X^\bullet$ the **multiple-point complex** of X.

Characteristic Polar Multiplicities and the Lê Numbers

- For any perverse sheaf $\mathbf{P}^\bullet$ on an open subset $\mathcal{U}$ of some $\mathbb{C}^N$, **the characteristic polar multiplicities of $\mathbf{P}^\bullet$** with respect to a “nice” choice of linear forms $\mathbf{z} = (z_0, \cdots, z_s)$, denoted $\lambda^i_{\mathbf{P}^\bullet, \mathbf{z}}(p)$ (defined in [Massey '94]) are non-negative integer-valued functions that mimic the purpose of the **Lê numbers** associated to non-isolated hypersurface singularities.

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- Indeed, for $0 \leq i \leq \dim_{\mathbf{0}} \Sigma f$, and all p in $\mathcal{U}$, one has the equalities $\lambda^i_{f,\mathbf{z}}(p) = \lambda^i_{\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[N]}(p)$.

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- **Example:** If $\dim_{\mathbf{0}} \Sigma f = 0$, $\lambda^0_{f,z}(\mathbf{0}) = \mu_{\mathbf{0}}(f)$ is the **Milnor number of f.**

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- **Example:** If $\dim_{\mathbf{0}} \Sigma f = 1$ and $\dim_{\mathbf{0}} \Sigma \left(f_{|_{V(z)}} \right) = 0$,

$$\lambda^1_{f,z}(\mathbf{0}) = \sum_{C \subseteq \Sigma f \text{ irred.}} \overset{\circ}{\mu}_C \left(C \cdot V(z) \right)_{\mathbf{0}},$$

where $\overset{\circ}{\mu}_C$ denotes the generic transverse Milnor number of f

Deforming a Parameterized Hypersurface

We recall the well-known result of [Milnor, 1968], relating the Milnor number $\mu_{\mathbf{0}}(f_0)$ of a plane curve singularity to the number of double points δ which occur in a generic (stable) deformation of f_0 by

$$\mu_{\mathbf{0}}(f_0) = 2\delta - r + 1,$$

where r is the number of irreducible components of the curve $V(f_0)$ at $\mathbf{0}$.

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We wish to generalize this formula, in light of recent work in [H., Massey 2017], in which we obtain a quick proof of the above formula, using the multiple-point complex $\mathbf{N}^\bullet_X$.

Deforming a Parameterized Hypersurface

The first question we ask is: **what if we didn't have such a "stable deformation" of the plane curve $V(f_0)$?** That is, what if we didn't know that the origin $\mathbf{0} \in V(f_0)$ splits into δ nodes? We can still use the techniques of [H.,Massey 2017] in this situation.

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$$\mu_{\mathbf{0}}(f_0) = -r + 1 + \sum_{p \in B_\epsilon \cap V(t - t_0)} \left(\mu_p(f_{t_0}) + m(p)\right)$$

where $m(p) := |\pi^{-1}(p)| - 1$.

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where $m(p) := |\pi^{-1}(p)| - 1$. But $m(p) = \operatorname{rank} H^0(\mathbf{N}^\bullet_X)_p$; we find then that, if we let $\mathbf{N}^\bullet_{X_{t_0}} = \mathbf{N}^\bullet_{X|_{V(t-t_0)}}[-1]$, then

$$\lambda^0_{\mathbf{N}^\bullet_{X_0},z}(\mathbf{0}) = r - 1, \text{ and } \lambda^0_{\mathbf{N}^\bullet_{X_{t_0}},z}(p) = m(p).$$

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Theorem (H., 2017)

Suppose that $\pi : (\mathcal{W}, \{0\} \times S) \to (\mathcal{U}, \mathbf{0})$ *is a one-parameter unfolding with parameter* t*, with* $\operatorname{im} \pi = V(f)$ *for some* $f \in \mathcal{O}^{\text{anal}}_{\mathcal{U},\mathbf{0}}$*. Suppose further that* $\mathbf{z} = (z_1, \cdots, z_n)$ *is chosen such that* $\mathbf{z}$ *is an IPA-tuple for* $f_0 = f_{|_{V(t)}}$ *at* $\mathbf{0}$*. Then, the following relationship holds for the Lê numbers of* f_0 *and the characteristic polar multiplicities of* $\mathbf{N}^\bullet_{X_{t_0}} := \mathbf{N}^\bullet_{X_{|_{V(t-t_0)}}}[-1]$ *with respect to* $\mathbf{z}$ *at* $\mathbf{0}$*: for* $0 < |t_0| \ll \epsilon \ll 1$ *and* $0 \le i \le n-2$*:*

$$\lambda^i_{f_0,\mathbf{z}}(\mathbf{0}) + \lambda^i_{\mathbf{N}^\bullet_{X_0},\mathbf{z}}(\mathbf{0}) = \sum_{p \in B_\epsilon \cap V(t-t_0, z_1, z_2, \cdots, z_i)} \left(\lambda^i_{f_{t_0},\mathbf{z}}(p) + \lambda^i_{\mathbf{N}^\bullet_{X_{t_0}},\mathbf{z}}(p) \right)$$

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- One immediately notices that the natural surjection $\mathbb{Q}^{\bullet}_{X}[n] \to \mathbf{I}^{\bullet}_{X}$ is an isomorphism precisely when $\mathbf{N}^{\bullet}_{X} = 0$; that is, the LCI X is a $\mathbb{Q}$-**homology manifold** precisely when the complex $\mathbf{N}^{\bullet}_{X}$ vanishes ([Borho, MacPherson]).

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- It is then natural to ask that, given the normalization Y of X and the resulting fundamental short exact sequence, **is there a similar result relating $\mathbf{N}_X^\bullet$ to whether or not Y is a $\mathbb{Q}$-homology manifold?**

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Theorem (H., 2018)

Y is a $\mathbb{Q}$-homology manifold if and only if $\mathbf{N}^\bullet_X$ has stalk cohomology concentrated in degree $-n+1$; i.e., for all $p \in X$, $H^k(\mathbf{N}^\bullet_X)_p$ is non-zero only possibly when $k = -n+1$.

Sketch of Proof ($\Longrightarrow$)

- Suppose first that the normalization Y is a $\mathbb{Q}$-homology manifold, and let $p \in X$ be arbitrary. Taking stalk cohomology at p yields the short exact sequence

$$0 \to \mathbb{Q} \to H^{-n}(\pi_* \mathbf{I}^\bullet_Y)_p \to H^{-n+1}(\mathbf{N}^\bullet_X)_p \to 0,$$

together with isomorphisms

$$H^k(\mathbf{N}^\bullet_X)_p \cong H^{k-1}(\pi_* \mathbf{I}^\bullet_Y)_p \cong \bigoplus_{q \in \pi^{-1}(p)} H^{k-1}(\mathbf{I}^\bullet_Y)_q$$

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- Since Y is a $\mathbb{Q}$-homology manifold, $\mathbb{Q}^\bullet_Y[n] \cong \mathbf{I}^\bullet_Y$, implying in particular that the stalk cohomology of $\mathbf{I}^\bullet_Y$ is concentrated in degree $-n$. **Hence, the stalk cohomology of $\mathbf{N}^\bullet_X$ is concentrated in degree $-n+1$.**

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- Suppose now that, for all $p \in X$, $H^k(\mathbf{N}^\bullet_X)_p$ is non-zero only possibly when $k = -n + 1$. We wish to show that **the natural morphism $\mathbb{Q}^\bullet_Y[n] \to \mathbf{I}^\bullet_Y$ is an isomorphism in $D^b_c(Y)$.**

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- It is immediately clear, from the isomorphisms $H^k(\pi_*\mathbf{I}^\bullet_Y)_p \cong H^{k+1}(\mathbf{N}^\bullet_X)_p$ for $-n+1 \leq k \leq -1$, that the stalk cohomology of $\mathbf{I}^\bullet_Y$ is concentrated in degree $-n$.

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- Consequently, the morphism $\mathbb{Q}^\bullet_Y[n] \to \mathbf{I}^\bullet_Y$ is an isomorphism in $D^b_c(Y)$ if and only if the map

$$H^{-n}(\mathbb{Q}^\bullet_Y[n])_q \cong \mathbb{Q} \to \mathbb{Q} \cong H^{-n}(\mathbf{I}^\bullet_Y)_q$$

is not the zero map.

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  is not the zero map. But this is just the "diagonal" morphism from a single copy of $\mathbb{Q}$ to the number of connected components of $Y \backslash \{p\}$, which is clearly non-zero. **Thus, Y is a $\mathbb{Q}$-homology manifold.**

$\mathbb{Q}$-Parameterizations

Let $\pi_0 : Y_0 \to X_0$ be the normalization of a hypersurface $X_0 = V(f_0)$, and suppose Y_0 is a $\mathbb{Q}$-homology manifold. Let $\pi : Y_0 \times \mathbb{C} \to X$ be a one-parameter unfolding of π_0. Then, we have the following theorem:

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Theorem (H.,2018)

Everything works exactly the same as in the case of a parameterization with smooth domain, and the same relationship holds between the characteristic polar multiplicities of $\mathbf{N}^\bullet_X$ *and* $\phi_f[-1]\mathbb{Q}^\bullet_{\mathcal{U}}[n+1]$.

Example

Let $f(x, y, z) = xz^2 - y^2(y + x^3)$, so that $X = V(f) \subseteq \mathbb{C}^3$ has $\Sigma f = V(y, z)$. Then, if

$$Y = V(u^2 - x(y + x^3), uy - xz, uz - y(y + x^3)) \subseteq \mathbb{C}^4,$$

the projection map $\pi : Y \to X$ **is the normalization of** X**.** It is easy to check that $\Sigma Y = V(x, y, z, u)$, and

$$\pi^{-1}(\Sigma f) = V(u^2 - x^4, y, z).$$

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the projection map $\pi : Y \to X$ is the normalization of X. It is easy to check that $\Sigma Y = V(x, y, z, u)$, and

$$\pi^{-1}(\Sigma f) = V(u^2 - x^4, y, z).$$

Let

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It then follows that $X_k = \emptyset$ if $k > 2$, and $X_2 = V(y, z) \backslash \{\mathbf{0}\}$, so that

$$\operatorname{supp} \mathbf{N}_X^\bullet = V(y, z) = \Sigma f.$$

Example

- We have the short exact sequence

$$0 \to \mathbb{Q} \to H^{-2}(\pi_* \mathbf{I}^\bullet_Y)_p \to H^{-1}(\mathbf{N}^\bullet_Y)_p \to 0$$

and isomorphism $H^{-1}(\pi_* \mathbf{I}^\bullet_Y)_p \cong H^0(\mathbf{N}^\bullet_Y)_p$.

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- We can then see that verifying the stalk cohomology of $\mathbf{N}^\bullet_X$ is concentrated in degree -1 is equivalent to verifying $H^{-1}(\pi_* \mathbf{I}^\bullet_Y)_p = 0$ for all $p \in X$.
- Conversely, since $\mathbf{I}^\bullet_{Y|_{Y \setminus \Sigma Y}} \cong \mathbb{Q}^\bullet_{Y \setminus \Sigma Y}[2]$, Y is a $\mathbb{Q}$-homology manifold if the stalk cohomology of $\mathbf{I}^\bullet_Y$ at $\mathbf{0} \in Y$ is non-zero only in degree -2, where it is of dimension one.

Example

- At $\mathbf{0} \in Y$, we find

$$H^{-2}(\mathbf{I}^\bullet_Y)_\mathbf{0} \cong \mathbb{H}^{-2}(K_{Y,\mathbf{0}}; \mathbf{I}^\bullet_Y) \cong H^1(K_{Y,\mathbf{0}}; \mathbb{Q}),$$

where $K_{Y,\mathbf{0}} = Y \cap S_\epsilon$ (for $0 < \epsilon \ll 1$) is the **real link** of Y at $\mathbf{0}$.

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- Since Y has an isolated singularity at the origin, the link $K_{Y,\mathbf{0}}$ is a compact, orientable, smooth manifold of real dimension 3. Hence, $H^0(K_{Y,\mathbf{0}}; \mathbb{Q}) \cong H^3(K_{Y,\mathbf{0}}; \mathbb{Q}) \cong \mathbb{Q}$.

Example

- The standard parameterization of the twisted cubic in $\mathbb{P}^3$ lifts to a parameterization of Y, which is isomorphic to the affine cone over the twisted cubic. This parameterization is 3-to-1, from which it follows (with some work) that we have a 3-to-1 cover of $K_{Y,\mathbf{0}}$ by the 3-sphere in $\mathbb{C}^2$. Hence,

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By the Universal Coefficient Theorem and Poincaré Duality,

$$H^2(K_{Y,\mathbf{0}};\mathbb{Q}) \cong H^1(K_{Y,\mathbf{0}};\mathbb{Q}) = 0,$$

so that Y is a $\mathbb{Q}$-homology manifold.

$\mathbf{N}^\bullet_X$ in the Literature

- When $X = V(f) \subseteq \mathcal{U} \subseteq \mathbb{C}^{n+1}$ is a hypersurface, [D. Massey, 2018] has recently shown that

$$\mathbf{N}^\bullet_X \cong \mathrm{Ker}\{\mathrm{id} - \widetilde{T}_f\},$$

where $\widetilde{T}_f$ is the monodromy automorphism on the vanishing cycles $\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]$, and the kernel is taken in the category $Perv(V(f))$.

- When X is a reduced complex algebraic variety of pure dimension n, [M. Saito, 2018] has recently shown that

$$W_0 H^1(X;\mathbb{Q}) \cong \mathrm{Coker}\{H^0(Y;\mathbb{Q}) \to \mathbb{H}^{-n+1}(X;\mathbf{N}^\bullet_X)\}$$

where $W_0 H^1(X;\mathbb{Q})$ denotes the weight zero part of the cohomology $H^1(X;\mathbb{Q})$, considered as a mixed Hodge structure, and Y is the normalization of X.

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  Problem: Need to understand.

Thank You!

Appendix: IPA-Deformations

In order to compute the characteristic polar multiplicities and Lê numbers, we need to choose linear forms that "cut down" the support in a certain way. We now give several equivalent conditions for this "cutting" procedure.

Proposition

f **is a deformation of** $f_{|_{V(L)}}$ **with isolated polar activity at 0** *if any of the equivalent hold:*

1. $\dim_{\mathbf{0}} \Gamma^1_{f,L} \cap V(L) \leq 0.$
2. $\dim_{(\mathbf{0},d_{\mathbf{0}}L)} \operatorname{im} dL \cap (f \circ \pi)^{-1}(0) \cap \overline{T^*_f \mathcal{U}}$, *where* $\pi : T^*\mathcal{U} \to \mathcal{U}$ *is the canonical projection map.*
3. $\dim_{(\mathbf{0},d_{\mathbf{0}}L)} SS(\psi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]) \cap \operatorname{im} dL \leq 0.$
4. $\dim_{(\mathbf{0},d_{\mathbf{0}}L)} SS(\mathbb{Z}^\bullet_{V(f)}[n]) \cap \operatorname{im} dL \leq 0.$
5. $\dim_{\mathbf{0}} \operatorname{supp} \phi_L[-1]\mathbb{Z}^\bullet_{V(f)}[n] \leq 0.$

Appendix: Characteristic Polar Multiplicities [Massey, 1994]

- Given a perverse sheaf $\mathbf{P}^\bullet$ on a complex analytic subset X of $\mathbb{C}^N$, and choice of "nice" tuple of linear forms $\mathbf{z} = (z_0, \ldots, z_s)$ on $\mathbb{C}^N$ (where $\dim \operatorname{supp} \mathbf{P}^\bullet = s$), the **characteristic polar multiplicities of $\mathbf{P}^\bullet$ with respect to $\mathbf{z}$** at a point $p \in X$ are the non-negative integers

  $$\lambda^i_{\mathbf{P}^\bullet, \mathbf{z}}(p) = \operatorname{rank}_{\mathbb{Z}} H^0(\phi_{z_i - z_i(p)}[-1]\psi_{z_{i-1}-z_{i-1}(p)}[-1] \cdots \psi_{z_0 - z_0(p)}[-1]\mathbf{P}$$

  for $0 \leq i \leq s$.
- Such numbers exist more generally for objects of $D^b_{\mathbb{C}-c}(X)$, but they are slightly more cumbersome to define (and no longer need to be non-negative!)
- Why are these useful? For all $p \in X$, one has

  $$\chi(\mathbf{P}^\bullet)_p = \sum_{i=0}^{s} (-1)^i \lambda^i_{\mathbf{P}^\bullet, \mathbf{z}}(p).$$

Appendix: Proof of Relationship

A large part of the proof is verifying that the IPA-condition is sufficient to simultaneously guarantee the numbers $\lambda^i_{f,(t,\mathbf{z})}(\mathbf{0})$ and $\lambda^i_{\mathbf{N}^\bullet_X,(t,\mathbf{z})}(\mathbf{0})$ are defined.

After applying the dynamic intersection property to rewrite $\lambda^i_{f_0,\mathbf{z}}(\mathbf{0})$ as a sum in the $t \neq 0$ slice, it suffices to prove

$$\lambda^0_{\mathbf{N}^\bullet_X,(t,\mathbf{z})} = -\lambda^0_{\mathbf{N}^\bullet_{X_0},\mathbf{z}}(\mathbf{0}) + \sum_{p \in B_\epsilon \cap V(t-t_0)} \lambda^0_{\mathbf{N}^\bullet_{X_{t_0}},\mathbf{z}}(p).$$

Since $(t,\mathbf{z})$ is an IPA-tuple for f at $\mathbf{0}$, we have

$$\lambda^0_{\mathbf{N}^\bullet_{X_0},\mathbf{z}}(\mathbf{0}) = \lambda^1_{\mathbf{N}^\bullet_X,(t,\mathbf{z})}(\mathbf{0}) - \lambda^0_{\mathbf{N}^\bullet_X,(t,\mathbf{z})}(\mathbf{0}).$$

The claim follows from again applying the dynamic intersection property for proper intersections with $\Lambda^1_{\mathbf{N}^\bullet_X,(t,\mathbf{z})}$.

Appendix: $\mathbf{N}_X^\bullet$ and Small Maps Example

$f(x, y, z, w) = xw - yz$, $X = V(f) \subseteq \mathbb{C}^4$, and set $Y = V(xt_0 + yt_1, zt_0 + wt_1) \subseteq \mathbb{C}^4 \times \mathbb{P}^1$. Then, the projection $\pi : Y \to X$ is a small map, where Y is smooth, and one-to-one away from $\mathbf{0}$ in X.

We then find:

$$H^k(\mathbf{N}_X^\bullet)_p = 0$$

for all $p \neq \mathbf{0}$ and all $k \in \mathbb{Z}$. At $\mathbf{0}$, we find:

$$H^k(\mathbf{N}_X^\bullet)_\mathbf{0} \cong \widetilde{H}^{k+2}(\mathbb{P}^1; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0 \\ 0, & \text{else} \end{cases}$$