# DEFORMATION FORMULAS FOR PARAMETERIZABLE HYPERSURFACES

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Abstract. We investigate one-parameter deformations of functions on affine space which define parameterizable hypersurfaces. With the assumption of isolated polar activity at the origin, we are able to completely express the Lê numbers of the special fiber in terms of the Lê numbers of the generic fiber and the characteristic polar multiplicities of the multiple-point complex, a perverse sheaf naturally associated to any parameterized hypersurface.

## 1. Milnor's Classical Result for Plane Curves

Suppose that $\mathcal{U}$ is an open neighborhood of the origin in $\mathbb{C}^2$. Let $g_0 : (\mathcal{U}, \mathbf{0}) \to (\mathbb{C}, 0)$ be a complex analytic function which has an isolated critical point at the origin. Thus, $g_0$ defines a plane curve $V(g_0)$ in $\mathcal{U}$. Let $r$ be the number of irreducible components of $V(g_0)$ at the origin. Then, by a well-known result of Milnor (Theorem 10.5 of [23]), the Milnor number $\mu_{\mathbf{0}}(g_0)$ is related to the number of double points $\delta$ which occur in a generic (stable) deformation of $g_0$ by

$$\mu_{\mathbf{0}}(g_0) = 2\delta - r + 1. \quad (\dagger)$$

We wish to generalize this formula, in light of recent work with the author and D.B. Massey in [8], in which we obtain a quick proof of the above formula.

The first question we ask is: **what if we didn't have such a "stable deformation" of $V(g_0)$?** That is, what if we didn't know that the origin $\mathbf{0} \in V(g_0)$ splits into $\delta$ nodes? We can still use the techniques of Theorem 5.3 of [8] in this situation. In this case, if $F$ is a finite, generically one-to-one morphism (as below) which parameterizes the deformation of $V(g_0)$, we have

$$\mu_{\mathbf{0}}(g_0) = -r + 1 + \sum_{p \in B_\epsilon \cap V(t - t_0)} (\mu_p(g_{t_0}) + m(p))$$

where $m(p) := |F^{-1}(p)| - 1$. One of the main ideas behind generalizing Milnor's formula to higher dimensions is that, when the hypersurface $V(g)$ is parameterized, there is a natural perverse sheaf on $V(g)$ that generalizes the function $m(p)$.

Suppose now that $\mathcal{U}$ is an open neighborhood of the origin in $\mathbb{C}^n$, $\mathbb{D}^\circ$ is an open disk around the origin in $\mathbb{C}$, and $g : (\mathbb{D}^\circ \times \mathcal{U}, \mathbf{0}) \to (\mathbb{C}, 0)$ is a reduced complex analytic function. Thus, $g$ defines a one-parameter analytic family $g_t(\mathbf{z}) := g(t, \mathbf{z})$. Additionally, assume $F : (\mathbb{D}^\circ \times \mathcal{W}, \mathbb{D}^\circ \times S) \to (\mathbb{D}^\circ \times \mathcal{U}, \mathbf{0})$ is a finite morphism that is generically one-to-one with $\operatorname{im} F = V(g)$ (i.e., $F$ is a parameterization of the "total" hypersurface $V(g)$), and $S$ is a finite subset of $\mathcal{W}$, an open subset of

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$\mathbb{C}^{n-1}$. For ease of notation, we frequently assume $\mathcal{W}$ and $\mathcal{U}$ are of the form $\mathbb{D}^\circ \times \widetilde{\mathcal{W}}$ or $\mathbb{D}^\circ \times \widetilde{\mathcal{U}}$ when considering one-parameter unfoldings of parameterizations. See [8] and Section 3.

What would it mean to have a generalization of (†)? In the broadest sense, one would want to express numerical data about the singularities of $g_0$ completely in terms of data about the singularities of $g_{t_0}$, for $t_0$ small and non-zero. What changes when we move to higher dimensions?

One of the restrictions in considering parameterizable hypersurfaces $V(g)$ is that they must have codimension-one singularities.

**Lemma 1.1.** *Suppose* $\operatorname{im} F = V(g)$ *is a parameterized hypersurface germ in* $(\mathbb{C}^{n+1}, \mathbf{0})$*, as above. Then, the singular locus* $\Sigma V(g)$ *of* $V(g)$ *has codimension one in* $V(g)$*.*

*Proof.* The existence of such a morphism $F$ is equivalent to assuming that $V(g)$ has a smooth normalization. Indeed, $F$ is a finite morphism from affine space, which is clearly normal. If the singular locus of $V(g)$ has codimension greater than one, $V(g)$ would already be a normal space, and be equal to its normalization. Since normalizations are unique, this is impossible. □

Consequently, since we must have codimension-one singularities, we may no longer use the Milnor number in higher dimensions. One natural generalization of the Milnor number to higher-dimensional singularities are the **Lê numbers** $\lambda^i_{g,\mathbf{z}}$, and we will express the Lê numbers of the $t=0$ slice of in terms of the Lê numbers of the $t \neq 0$ slice, together with the **characteristic polar multiplicities** of $\mathbf{N}^\bullet_F$, a certain perverse sheaf associated to parameterized hypersurfaces. This will be explored in Section 3 and Section 4.

When moving to higher dimensions, we must also consider which sort of deformation to allow when relating $g_0$ and $g_{t_0}$ for $t_0$ small and not zero. For this, we choose the notion of an **IPA-deformation**; these are deformations which, intuitively, are those where the only "interesting" behavior happens at the origin, and the only change propagates outwards from the origin along curves. Such deformations exist generically in all dimensions. We examine this notion, first introduced by Massey in [14], in Section 2.

In Section 5, we prove the following result.

**Main Theorem** (5.2)**.** *Suppose that* $F : (\mathcal{W}, \{0\} \times S) \to (\mathcal{U}, \mathbf{0})$ *is a one-parameter unfolding with parameter* $t$*, with* $\operatorname{im} F = V(g)$ *for some* $g \in \mathcal{O}^{\text{anal}}_{\mathcal{U},\mathbf{0}}$*. Suppose further that* $\mathbf{z} = (z_1, \cdots, z_n)$ *is chosen such that* $\mathbf{z}$ *is an IPA-tuple for* $g_0 = g_{|_{V(t)}}$ *at* $\mathbf{0}$*. Then, the following formulas hold for the Lê numbers of* $g_0$ *with respect to* $\mathbf{z}$ *at* $\mathbf{0}$*: for* $0 < |t_0| \ll \epsilon \ll 1$*,*

$$\lambda^0_{g_0,\mathbf{z}}(\mathbf{0}) = -\lambda^0_{\mathbf{N}^\bullet_{f_0},\mathbf{z}}(\mathbf{0}) + \sum_{p \in B_\epsilon \cap V(t-t_0)} \left( \lambda^0_{g_{t_0},\mathbf{z}}(p) + \lambda^0_{\mathbf{N}^\bullet_{f_{t_0}},\mathbf{z}}(p) \right),$$

*and, for* $1 \leq i \leq n-2$*,*

$$\lambda^i_{g_0,\mathbf{z}}(\mathbf{0}) = \sum_{q \in B_\epsilon \cap V(t-t_0, z_1, z_2, \cdots, z_i)} \lambda^i_{g_{t_0},\mathbf{z}}(q).$$

*In particular, the following relationship holds for* $0 \leq i \leq n-2$*:*

$$\lambda^i_{g_0,\mathbf{z}}(\mathbf{0}) + \lambda^i_{\mathbf{N}^\bullet_{f_0},\mathbf{z}}(\mathbf{0}) = \sum_{p \in B_\epsilon \cap V(t-t_0, z_1, z_2, \cdots, z_i)} \left( \lambda^i_{g_{t_0},\mathbf{z}}(p) + \lambda^i_{\mathbf{N}^\bullet_{f_{t_0}},\mathbf{z}}(p) \right)$$

## 2. IPA-Deformations

Although we need to consider only the case of a family of parameterized hypersurfaces for this paper, much of the machinery we use for Section 4 and Section 5 does not require such restrictive hypotheses. That is, the notion of IPA-deformations and Lê numbers (see Massey, [14] and [15]) apply to hypersurface singularities in general, not just parameterized hypersurfaces.

Suppose $\mathbf{z} = (z_0, \cdots, z_n)$ are local coordinates on an open neighborhood $\mathcal{U} \subseteq \mathbb{C}^{n+1}$ of $\mathbf{0}$, so that we have $\mathcal{T}^*\mathcal{U} \cong \mathcal{U} \times \mathbb{C}^{n+1}$, with fiber-wise basis $(d_p z_0, \cdots, d_p z_n)$ of $(T^*\mathcal{U})_p = \pi^{-1}(p)$, where $\pi : T^*\mathcal{U} \to \mathcal{U}$ is the canonical projection map.

Denote by $\mathrm{Span}\langle dz_0, \cdots, dz_k \rangle$ the subset of $T^*\mathcal{U}$ given by $\{(p, \sum_{i=0}^k w_i d_p z_i) \,|p \in \mathcal{U}, w_i \in \mathbb{C}\}$

Let $g : (\mathcal{U}, \mathbf{0}) \to (\mathbb{C}, 0)$ be a (reduced) complex analytic function, where $\mathcal{U}$ is a connected open neighborhood of the origin in $\mathbb{C}^{n+1}$.

Finally, let $\overline{T^*_g\mathcal{U}}$ denote the (closure of) the relative conormal space of $g$ in $\mathcal{U}$, i.e.,

$$\overline{T^*_g\mathcal{U}} := \overline{\{(p, \xi) \in T^*\mathcal{U} \,|\, \xi(\ker d_p g) = 0\}}.$$

It is important to note that $\overline{T^*_g\mathcal{U}}$ is a $\mathbb{C}$-conic subset of $T^*\mathcal{U}$, as we will consider its projectivization in Definition 2.2.

The following definitions of the relative polar varieties of $g$ differ slightly from their more classical construction (see, for example [7], [11], or [10]), following that of [14],[16]. Lastly, the intersection product appearing in the following definitions is that of proper intersections in complex manifolds (See Chapter 6 of [4]).

**Definition 2.1.** *The* ***relative polar curve of*** $g$ ***with respect to*** $\mathbf{z}$*, denoted* $\Gamma^1_{g,\mathbf{z}}$*, is, as an analytic cycle at the origin, the collection of those components of the cycle*

$$\pi\left(\overline{T^*_g\mathcal{U}} \cdot \mathrm{im}\ dz_0\right)$$

*which are not contained in* $\Sigma g$*, provided that* $\overline{T^*_g\mathcal{U}}$ *and* $\mathrm{im}\ dz_0$ *intersect properly in* $T^*\mathcal{U}$*.*

More generally, one can define the higher $k$-dimensional relative polar varieties $\Gamma^k_{g,\mathbf{z}}$ in this manner, by considering the *projectivized* relative conormal space $\mathbb{P}(\overline{T^*_g\mathcal{U}})$ as follows. For $0 \leq k \leq n$, consider the subspace $\mathbb{P}(\mathrm{Span}\langle dz_0, \cdots, dz_k\rangle)$ of $\mathbb{P}(T^*\mathcal{U}) \cong \mathcal{U} \times \mathbb{P}^n$, the projectivized cotangent bundle of $\mathcal{U}$.

**Definition 2.2.** *The* ***$k$-dimensional relative polar variety of*** $g$ ***with respect to*** $\mathbf{z}$*, denoted* $\Gamma^k_{g,\mathbf{z}}$*, is, as an analytic cycle at the origin, the collection of those components of*

$$\pi\left(\mathbb{P}(\overline{T^*_g\mathcal{U}}) \cdot \mathbb{P}\left(\mathrm{Span}\langle dz_0, \cdots, dz_k\rangle\right)\right)$$

*which are not contained in the critical locus* $\Sigma g$ *at the origin, provided that* $\mathbb{P}(\overline{T^*_g\mathcal{U}})$ *and* $\mathbb{P}\left(\mathrm{Span}\langle dz_0, \cdots, dz_k\rangle\right)$ *intersect properly in* $T^*\mathcal{U}$*. By abuse of notation, we also use* $\pi$ *to denote the canonical projection* $\mathbb{P}(T^*\mathcal{U}) \to \mathcal{U}$*.*

See Definition 7.1 in the appendix for the classical definition of $\Gamma^k_{g,\mathbf{z}}$.

**Remark 2.3.** When considering the relative polar curve, we frequently write $\Gamma^1_{g,z_0}$ in place of $\Gamma^1_{g,\mathbf{z}}$, to highlight the fact that the relative polar curve depends only on the function $g$ and choice of a single linear form $z_0$.

Throughout this paper, we will use the (shifted) **nearby and vanishing cycle functors** $\psi_g[-1]$ and $\phi_g[-1]$, respectively, from the bounded derived category $D^b_c(\mathcal{U})$ of constructible complexes of sheaves on $\mathcal{U}$ to those on $V(g)$ (see for example [9], [3], [6], or [1]) .

We will also make frequent use of the **microsupport** of a (bounded, constructible) complex of sheaves $\mathbf{F}^\bullet$, denoted $SS(\mathbf{F}^\bullet)$, is a closed $\mathbb{C}^\times$-conic subset of $T^*\mathcal{U}$. We will use the following characterization of $SS(\mathbf{F}^\bullet)$ in terms the vanishing cycles (See Prop 8.6.4, of [9]).

**Proposition 2.4** (Microsupport)**.** *Let* $\mathbf{F}^\bullet \in D^b_c(\mathcal{U})$ *and let* $(p,\xi) \in T^*\mathcal{U}$*. Then, the following are equivalent:*

*(1)* $(p,\xi) \notin SS(\mathbf{F}^\bullet)$.
*(2) There exists an open neighborhood* $\Omega$ *of* $(p,\xi)$ *in* $T^*\mathcal{U}$ *such that, for any* $q \in \mathcal{U}$ *and any complex analytic function* $f$ *defined in a neighborhood of* $q$ *with* $f(q) = 0$ *and* $(q, d_q f) \in \Omega$*, one has* $(\phi_f \mathbf{F}^\bullet)_q = 0$.

In order to compute numerical invariants associated to certain perverse sheaves (see the characteristic polar multiplicities Section 4 and Lê numbers), we need to choose linear forms that "cut down" the support in a certain way. We now give several equivalent conditions for this "cutting" procedure, that will be used throughout this paper.

**Proposition 2.5.** *The following are equivalent:*

*(1)* $\dim_{\mathbf{0}} \Gamma^1_{g,z_0} \cap V(z_0) \le 0$.
*(2)* $\dim_{\mathbf{0}} \Gamma^1_{g,z_0} \cap V(g) \le 0$.
*(3)* $\dim_{(\mathbf{0},d_{\mathbf{0}}z_0)} \mathrm{im}\ dz_0 \cap (g\circ\pi)^{-1}(0) \cap \overline{T^*_g\mathcal{U}}$*, where* $\pi : T^*\mathcal{U} \to \mathcal{U}$ *is the canonical projection map.*
*(4)* $\dim_{(\mathbf{0},d_{\mathbf{0}}z_0)} SS(\psi_g[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]) \cap \mathrm{im}\ dz_0 \le 0$.
*(5)* $\dim_{(\mathbf{0},d_{\mathbf{0}}z_0)} SS(\mathbb{Z}^\bullet_{V(g)}[n]) \cap \mathrm{im}\ dz_0 \le 0$.
*(6)* $\dim_{\mathbf{0}} \mathrm{supp}\, \phi_{z_0}[-1]\mathbb{Z}^\bullet_{V(g)}[n] \le 0$.
*(7) Away from* $\mathbf{0}$*, the comparison morphism* $\mathbb{Z}^\bullet_{V(g,z_0)}[n-1] \to \psi_{z_0}[-1]\mathbb{Z}^\bullet_{V(g)}[n]$ *is an isomorphism.*

*Proof.* The equivalence of statements (1), (2), and (3) are covered in Proposition 2.6 of [14].

The equivalence (3) $\iff$ (4) follows directly from the equality

$$\overline{T^*_g\mathcal{U}} \cap (g\circ\pi)^{-1}(0) = SS(\psi_g[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]).$$

(See [2] for the original result, although the phrasing used above is found in [18]).

To see the equivalence (4) $\iff$ (5), consider the natural distinguished triangle

$$i_*i^*[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1] \to j_!j^!\mathbb{Z}^\bullet_{\mathcal{U}}[n+1] \to \mathbb{Z}^\bullet_{\mathcal{U}}[n+1] \xrightarrow{+1} \quad (\ddagger)$$

where $i : V(g) \hookrightarrow \mathcal{U}$, and $j : \mathcal{U}\backslash V(g) \hookrightarrow \mathcal{U}$. Then, by [21], there is an equality of microsupports

$$SS(\psi_g[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]) = SS(j_!j^!\mathbb{Z}^\bullet_{\mathcal{U}}[n+1])_{\subseteq V(g)},$$

where the subscript $\subseteq V(g)$ denotes the union of irreducible components of $SS(j_!j^!\mathbb{Z}_{\mathcal{U}}[n+1])$ that lie over the hypersurface $V(g)$. But, since $SS(\mathbb{Z}_{\mathcal{U}}[n+1]) \cong \mathcal{U} \times \{\mathbf{0}\}$, ($\ddagger$) implies that

$$SS(i_*i^*[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]) = SS(j_!j^!\mathbb{Z}^\bullet_{\mathcal{U}}[n+1])_{\subseteq V(g)},$$

by the triangle inequality for microsupports. The claim follows after noting $i_*i^*[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1] = \mathbb{Z}^\bullet_{V(g)}[n]$.

The equivalence (5) $\iff$ (6) follows easily from Proposition 2.4, or see Theorem 3.1 of [13].

Lastly, one concludes (6) $\iff$ (7) trivially from the short exact sequence of perverse sheaves

$$0 \to \mathbb{Z}^\bullet_{V(g,z_0)}[n-1] \to \psi_{z_0}[-1]\mathbb{Z}^\bullet_{V(g)}[n] \to \phi_{z_0}[-1]\mathbb{Z}^\bullet_{V(g)}[n] \to 0$$

on $V(g, z_0)$.

□

**Definition 2.6.** *Given an analytic function $g : (\mathcal{U}, \mathbf{0}) \to (\mathbb{C}, 0)$ and a non-zero linear form $z_0 : (\mathcal{U}, 0) \to (\mathbb{C}, 0)$, we say that $g$ is a deformation of $g_{|_{V(z_0)}}$ with **isolated polar activity** at $\mathbf{0}$ (or, an **IPA deformation** for short) if the equivalent statements of Proposition 2.5 hold.*

We can iterate the notion of an IPA-deformation as follows.

**Definition 2.7.** *Let $k \geq 0$. A $(k+1)-$tuple $(z_0, \cdots, z_k)$ is said to be an IPA-tuple for $g$ at $\mathbf{0}$ if, for all $1 \leq i \leq k$, $g_{|_{V(z_0,\cdots,z_{i-1})}}$ is an IPA-deformation of $g_{|_{V(z_0,\cdots,z_i)}}$ at $\mathbf{0}$.*

The following lemma follows from an inductive application of Theorem 1.1 of [12], and is crucial for our understanding of what IPA-deformation "looks like" in the cotangent bundle (cf. Proposition 2.5, item (2)).

**Lemma 2.8** (Massey)**.** *Let $k \geq 0$. Then, for all $p \in V(z_0, \cdots, z_{k-1})$ with $d_pz_k \notin \left(\overline{T^*_{g_{|_{V(z_0,\cdots,z_{k-1})}}}V(z_0,\cdots,z_{k-1})}\right)_p$, we have*

$$\left(\overline{T^*_g\mathcal{U}}\right)_p \cap \operatorname{Span}\langle d_pz_0, \cdots, d_pz_k\rangle = 0.$$

**Remark 2.9.** We want to briefly explain the motivation of Lemma 2.8. If $g$ is an IPA-deformation of $g_{|_{V(z_0)}}$ at $\mathbf{0}$, then for all $p \in V(g, z_0)$ away from $\mathbf{0}$ we know by Proposition 2.5 that we have

$$d_pz_0 \notin SS(\psi_g[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]).$$

That is, the cohomology of $\psi_g[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]$ is unchanged when one moves in the direction of $dz_0$ from points $p$ away from the origin.

Although it takes a lot more familiarity with the derived category, one can further show that, at such a point $p$, one has a isomorphism

$$\psi_{g_{|_{V(z_0)}}}[-1]\mathbb{Z}^\bullet_{V(z_0)}[n] \cong \psi_{z_{0|_{V(g)}}}[-1]\psi_g[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]$$

(see Proposition 3.1 of [12]). Strikingly, this implies the equality

$$SS(\psi_g[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1])\cap((g,z_0)\circ\pi)^{-1}(0) = SS(\psi_{g_{|_{V(z_0)}}}[-1]\mathbb{Z}^\bullet_{V(z_0)}[n])\cap((g,z_0)\circ\pi)^{-1}(0)$$

over points $p \neq \mathbf{0}$.

The main goal of this section is the following result. This result, originally from [15], is presented here with the weaker hypothesis of choosing an IPA-tuple, in lieu of a prepolar-tuple. For the definition of the Lê numbers of $g$ with respect to a tuple of linear forms $\mathbf{z}$, see the appendix.

**Proposition 2.10.** *Suppose that $\mathbf{z} = (z_0, \cdots, z_n)$ is an IPA-tuple for $g$ at $\mathbf{0}$, and use coordinates $\widetilde{\mathbf{z}} = (z_1, \cdots, z_n)$ for $V(z_0)$. Then, for $0 \leq i \leq \dim_{\mathbf{0}} \Sigma g$, the Lê numbers $\lambda^i_{g,\mathbf{z}}(\mathbf{0})$ are defined, and the following equalities hold:*

$$\lambda^0_{g_{|_{V(z_0)}},\widetilde{\mathbf{z}}}(\mathbf{0}) = \left(\Gamma^1_{g,z_0} \cdot V(z_0)\right)_{\mathbf{0}} + \lambda^1_{g,\mathbf{z}}(\mathbf{0})$$
$$\lambda^i_{g_{|_{V(z_0)}},\widetilde{\mathbf{z}}}(\mathbf{0}) = \lambda^{i+1}_{g,\mathbf{z}}(\mathbf{0}),$$

*for $1 \leq i \leq \dim_{\mathbf{0}} \Sigma g - 1$, where $\Gamma^1_{g,z_0}$ is the relative polar curve of $g$ with respect to $z_0$.*

*Proof.* The proof follows Theorem 1.28 of [15], *mutatis mutandis* (changing prepolar to IPA).

Via the Chain Rule, it suffices to demonstrate that

$$\dim_{\mathbf{0}} \Gamma^{i+1}_{g,\mathbf{z}} \cap V(g) \cap V(z_0, \cdots, z_{i-1}) \leq 0,$$

since any analytic curve in $\Gamma^{i+1}_{g,\mathbf{z}} \cap V\left(\frac{\partial g}{\partial z_i}\right) \cap V(z_0, \cdots, z_{i-1})$ passing through $\mathbf{0}$ must be contained in $V(g)$, where $\Gamma^{i+1}_{g,\mathbf{z}}$ is the $(i+1)$-dimensional relative polar variety of $g$ with respect to $\mathbf{z}$.

Suppose that we had a sequence of points $p \in \Gamma^{i+1}_{g,\mathbf{z}} \cap V(g) \cap V(z_0, \cdots, z_{i-1})$ approaching $\mathbf{0}$. As each $p$ is contained in $\Gamma^{i+1}_{g,\mathbf{z}}$, for each $p$ we can find a sequence $p_k \to p$ with $p_k \notin \Sigma g$ satisfying $\langle d_{p_k} g\rangle \subseteq \operatorname{Span}\langle d_{p_k} z_0, \cdots, d_{p_k} z_{i-1}\rangle$ for each $k$. But then, by construction, we have found a nonzero element in the intersection $\left(\overline{T^*_g \mathcal{U}}\right)_p \cap \operatorname{Span}\langle d_p z_0 \cdots, d_p z_{i-1}\rangle$, contradicting Lemma 2.8.

□

## 3. Parameterizations and the Multiple-Point Complex

The second main object of this paper is a certain perverse sheaf, called the **multiple-point complex** of the parameterization, that naturally lives on any parameterized hypersurface $V(g)$. In order to define this object, we must first define parameterizations. The majority of this section follows [8].

**Definition 3.1.** *A* ***parameterization*** *of a hypersurface* $(V(g),\mathbf{0}) \subseteq (\mathcal{U},\mathbf{0})$ *is a finite morphism* $F : (\mathcal{W}, S) \to (\mathcal{U},\mathbf{0})$ *that is generically one-to-one, with* $\operatorname{im} F = V(g)$*, where* $\mathcal{W}$ *is an open subset of* $\mathbb{C}^n$ *(or, perhaps, a disjoint union of copies of* $\mathbb{C}^n$*), and* $S$ *is a finite subset of* $\mathcal{W}$*. We often implicitly restrict the codomain of a parameterization, and just write* $F : (\mathcal{W}, S) \to (V(g),\mathbf{0})$*.*

*A parameterization* $F : (\mathcal{W}, S) \to (V(g),\mathbf{0})$ *is said to be a* ***one-parameter unfolding*** *with unfolding parameter* $t$ *if* $F$ *is of the form*

$$F(t,\mathbf{z}) = (t, f_1(t,\mathbf{z}), \cdots, f_n(t,\mathbf{z}))$$

*where* $f_0(\mathbf{z}) := F(0,\mathbf{z})$ *is a generically one-to-one parameterization of* $V(g,t)$*.*

**Remark 3.2.** We recall that, as in Lemma 1.1, the existence of a parameterization $F$ for a hypersurface $V(g)$ is equivalent to assuming $V(g)$ has a smooth normalization, where $F$ is the normalization map.

We say that a parameterization $f_0$ has an **isolated instability** at $\mathbf{0}$ with respect to an unfolding $F$ of $f_0$ with parameter $t$ if one has $\dim_{\mathbf{0}} \Sigma_{\text{top}} t_{|_{\operatorname{im} F}} \leq 0$, where

$$\Sigma_{\text{top}} t := \overline{\{x \in V(t) \cap X \mid M_{t,x} \text{ does not have the integral cohomology of a point}\}}.$$

(cf. [24]) For this definition of critical locus, see [13].

The following proposition is one of our main motivations for using IPA-deformations: they naturally appear from one-parameter unfoldings with isolated instabilities.

**Proposition 3.3.** *Suppose* $F : (\mathcal{W}, S) \to (\mathcal{U},\mathbf{0})$ *is a 1-parameter unfolding of* $f_0$ *with unfolding parameter* $t$*, such that* $f_0$ *has an isolated instability at* $\mathbf{0}$ *with respect to* $F$*. If* $\operatorname{im} F = V(g)$ *for some* $g \in \mathcal{O}^{\text{anal}}_{\mathcal{U},\mathbf{0}}$*, then* $g$ *is an IPA-deformation of* $g_{|_{V(t)}}$ *at* $\mathbf{0}$*.*

*Proof.* By definition, $f_0$ has an isolated instability at $\mathbf{0}$ with respect to the unfolding $F$ with parameter $t$ if

$$\dim_{\mathbf{0}} \Sigma_{\text{top}} \left( t_{|_{V(g)}} \right) \leq 0.$$

Following Definition 1.9 of [13],

$$\begin{aligned}\Sigma_{\text{top}} \left( t_{|_{V(g)}} \right) &= \overline{\{p \in V(g) \mid (p, d_p t) \in SS(\mathbb{Z}^\bullet_{V(g)}[n])\}} \\ &= \pi \left( SS(\mathbb{Z}^\bullet_{V(g)}[n]) \cap \operatorname{im} dt \right),\end{aligned}$$

where $\pi : T^*\mathcal{U} \to \mathcal{U}$ is the natural projection. This follows immediately from Proposition 2.4.

Consequently, if $\dim_{\mathbf{0}} \Sigma_{\text{top}} \left( t_{|_{V(g)}} \right) \leq 0$, it follows that $(\mathbf{0}, d_{\mathbf{0}} t)$ is an isolated point in the intersection $SS(\mathbb{Z}^\bullet_{V(g)}[n]) \cap \operatorname{im} dt$, and the the result follows by Proposition 2.5. □

In the Abelian category of perverse sheaves on $V(g)$, there is a canonical surjective morphism $\mathbb{Z}^\bullet_{V(g)}[n] \xrightarrow{c} F_* \mathbb{Z}^\bullet_{\mathcal{W}}[n] \cong F_* F^* \mathbb{Z}^\bullet_{V(g)}[n]$, from the shifted constant sheaf on $V(g)$ to the proper push-forward of the shifted constant sheaf on $\mathcal{W}$. Since $F$ is a **small resolution** in the sense of M. Goresky and R. MacPherson [5], $F_* \mathbb{Z}^\bullet_{\mathcal{W}}[n]$ is the intersection cohomology complex with constant coefficients on

$V(g)$. We let $\mathbf{N}_F^\bullet$ be the kernel of this morphism, so that we have a short exact sequence of perverse sheaves

$$0 \to \mathbf{N}_F^\bullet \to \mathbb{Z}_{V(g)}^\bullet[n] \xrightarrow{c} F_*\mathbb{Z}_{\mathcal{W}}^\bullet[n] \to 0.$$

**Remark 3.4.** The reader may be wondering why the morphism $c$ has a non-zero kernel in the category of perverse sheaves. After all, on the level of stalks, the map $c$ is the diagonal inclusion map; it may seem as though $c$ should have a non-trivial cokernel, not kernel.

It is true that there is a complex of sheaves $\mathbf{C}^\bullet$ and a distinguished triangle in the derived category

$$\mathbb{Z}_{V(g)}^\bullet[n] \xrightarrow{c} F_*\mathbb{Z}_{\mathcal{W}}^\bullet[n] \to \mathbf{C}^\bullet \xrightarrow{[1]} \mathbb{Z}_{V(g)}^\bullet[n]$$

in which the stalk cohomology of $\mathbf{C}^\bullet$ is non-zero only in degree $-n$ and, in degree $-n$, is isomorphic to the cokernel of map induced on the stalks by $c$. However, the complex $\mathbf{C}^\bullet$ is **not** perverse; it is supported on a set of dimension $n-1$ and has non-zero cohomology in degree $-n$.

However, we can "turn" the triangle to obtain a distinguished triangle

$$\mathbf{C}^\bullet[-1] \to \mathbb{Z}_{V(g)}^\bullet[n] \xrightarrow{c} F_*\mathbb{Z}_{\mathcal{W}}^\bullet[n] \xrightarrow{[1]} \mathbf{C}^\bullet,$$

where $\mathbf{C}^\bullet[-1]$ is, in fact, perverse. Thus, in the Abelian category of perverse sheaves $\mathbf{N}_F^\bullet := \mathbf{C}^\bullet[-1]$ is the kernel of the morphism $c$.

**Definition 3.5.** *This perverse sheaf $\mathbf{N}_F^\bullet$ is called the* ***multiple-point complex*** *of the parameterization $F$, and is supported on the* ***image multiple-point set*** $D := \overline{\{p \in V(g) \mid |F^{-1}(p)| > 1\}}$.

**Proposition 3.6.** *The multiple-point complex $\mathbf{N}_F^\bullet$ has several useful properties.*

*(1) There is a short exact sequence of perverse sheaves*

$$0 \to \mathbf{N}_F^\bullet \to \mathbb{Z}_{V(g)}^\bullet[n] \to F_*\mathbb{Z}_{\mathcal{W}}^\bullet[n] \to 0.$$

*(2) It has non-zero stalk cohomology only in degree $-(n-1)$, where $n = \dim_{\mathbf{0}} V(g)$.*

*(3) In degree $-(n-1)$, the stalk cohomology is very easy to describe: for $p \in V(g)$,*

$$H^{-(n-1)}(\mathbf{N}_F^\bullet)_{\mathbf{p}} \cong \mathbb{Z}^{m(\mathbf{p})},$$

*where $m(\mathbf{p}) = |F^{-1}(\mathbf{p})| - 1$.*

*(4) The image multiple-point set $D$ is purely $(n-1)$-dimensional at $\mathbf{0}$ and contained in $\Sigma V(g)$. So, $D$ is a union of irreducible components of $\Sigma V(g)$.*

*Proof.* Claims (1), (2), and (3) all follow quickly from the short exact sequence

$$0 \to \mathbf{N}_F^\bullet \to \mathbb{Z}_{V(g)}^\bullet[n] \to F_*\mathbb{Z}_{\mathcal{W}}^\bullet[n] \to 0$$

of perverse sheaves on $V(g)$ which defines $\mathbf{N}_F^\bullet$.

For (4), note that $D$ is the support of a perverse sheaf which, on an open dense subset of $D$, has non-zero stalk cohomology in degree $-(n-1)$, it follows that $D$ is purely $(n-1)$-dimensional.

Since $F_*\mathbb{Z}_{\mathcal{W}}^\bullet[n]$ is intersection cohomology on $V(g)$ with constant coefficients, we have $(F_*\mathbb{Z}_{\mathcal{W}}^\bullet[n])_{|_{V(g)\backslash\Sigma V(g)}} \cong \mathbb{Z}_{V(g)\backslash\Sigma V(g)}^\bullet[n]$; hence, $\mathbf{N}_{F|_{V(g)\backslash\Sigma V(g)}}^\bullet = 0$, and the claim follows. □

**Remark 3.7.** If $F$ is a one-parameter unfolding of a parameterization $f_0$, then for all $t_0$ small, it is easy to see that there is an isomorphism $\mathbf{N}^\bullet_{F|_{V(t-t_0)}}[-1] \cong \mathbf{N}^\bullet_{f_{t_0}}$, where $f_{t_0}(\mathbf{z}) = F(\mathbf{z}, t_0)$.

**Example 3.8.** In the situation of Milnor's double-point formula, $F : (\mathbb{C}^2, \{0\} \times S) \to (\mathbb{C}^3, \mathbf{0})$ parameterizes the deformation of the curve $V(g_0)$ with $r$ irreducible components at $\mathbf{0}$ into a curve $V(g_{t_0})$ with only double-point singularities. Hence, $\dim_{\mathbf{0}} V(g) = 2$, and the image multiple-point set $D$ is purely 1-dimensional at $\mathbf{0}$.

Since $F$ is a one-parameter unfolding with parameter $t$, we moreover have

$$\mathbf{N}^\bullet_{F|_{V(t-t_0)}}[-1] \cong \mathbf{N}^\bullet_{f_{t_0}},$$

where $\mathbf{N}^\bullet_{f_{t_0}}$ is the multiple-point complex of the parameterization $f_{t_0}(\mathbf{z})$. For all $t_0 \neq 0$ small, $\mathbf{N}^\bullet_{f_{t_0}}$ is supported on the set of double points of $V(g_{t_0})$, and at each such double-point $p$ we have $\operatorname{rank} H^0(\mathbf{N}^\bullet_{f_0})_p = |F^{-1}(p)| - 1 = 1$.

At $\mathbf{0} \in V(g_0)$, we have $F^{-1}(\mathbf{0}) = S$, and $|S| = r$ by assumption. Thus, $\operatorname{rank} H^0(\mathbf{N}^\bullet_{f_0})_{\mathbf{0}} = r - 1$.

## 4. Characteristic Polar Multiplicities

The central concept of this section, the characteristic polar multiplicities of a perverse sheaf, were first defined and explored in [17]. These multiplicities, defined with respect to a "nice" choice of a tuple of linear forms $\mathbf{z} = (z_0, \cdots, z_s)$, are non-negative integer-valued functions that mimic the construction of the Lê numbers associated to non-isolated hypersurface singularities (see [15]).

Recall that the (shifted) nearby and vanishing cycle functors take perverse sheaves to perverse sheaves, and at an isolated point in the support of a perverse sheaf, the only possible non-zero stalk cohomology is in degree zero.

**Definition 4.1.** *Let $\mathbf{P}^\bullet$ be a perverse sheaf on $V(g)$, with $\dim_{\mathbf{0}} \operatorname{supp} \mathbf{P}^\bullet = s$. Let $\mathbf{z} = (z_0, \cdots, z_s)$ be a tuple of linear forms such that, for all $0 \leq i \leq s$, we have*

$$\dim_{\mathbf{0}} \operatorname{supp} \phi_{z_i - z_i(p)}[-1]\psi_{z_{i-1}-z_{i-1}(p)}[-1] \cdots \psi_{z_0 - z_0(p)}[-1]\mathbf{P}^\bullet \leq 0.$$

*Then, the $i$-dimensional **characteristic polar multiplicity of** $\mathbf{P}^\bullet$ with respect to $\mathbf{z}$ at $p \in V(g)$ is given by the formula*

$$\lambda^i_{\mathbf{P}^\bullet, \mathbf{z}}(p) = \operatorname{rank}_{\mathbb{Z}} H^0(\phi_{z_i - z_i(p)}[-1]\psi_{z_{i-1}-z_{i-1}(p)}[-1] \cdots \psi_{z_0 - z_0(p)}\mathbf{P}^\bullet)_p.$$

**Remark 4.2.** For a given perverse sheaf $\mathbf{P}^\bullet$, we will often say that the $i$-dimensional characteristic polar multiplicity $\lambda^i_{\mathbf{P}^\bullet, \mathbf{z}}(p)$ is **defined** at a point $p$ provided that

$$\dim_{\mathbf{0}} \operatorname{supp} \phi_{z_i - z_i(p)}[-1]\psi_{z_{i-1}-z_{i-1}(p)}[-1] \cdots \psi_{z_0 - z_0(p)}[-1]\mathbf{P}^\bullet \leq 0.$$

**Remark 4.3.** In general, one can define the characteristic polar multiplicities of any object in the bounded, derived category of constructible sheaves on $V(g)$, but they are slightly more cumbersome to define, and no longer need to be non-negative.

**Example 4.4.** Let $f : \mathcal{U} \to \mathbb{C}$ be an analytic function, with $f(\mathbf{0}) = 0$, $\mathcal{U}$ an open neighborhood of the origin in $\mathbb{C}^{n+1}$, and $\dim_{\mathbf{0}} \Sigma f = s$. Then, $\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]$ is a perverse sheaf on $V(f)$, with support equal to $\Sigma f \cap V(f)$. Indeed, the containment $\operatorname{supp} \phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1] \subseteq \Sigma f \cap V(f)$ follows from the complex analytic Implicit Function Theorem, and the reverse containment follows from Corollary 7.3 of [23] (for non-isolated singularities, this holds by A'Campo's result that, at a critical point, the Lefschetz number of the monodromy of the Milnor fiber is zero).

We then have

$$\lambda^i_{f,\mathbf{z}}(p) = \lambda^i_{\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1],\mathbf{z}}(p)$$

for all $0 \leq i \leq s$, and all $p$ in an open neighborhood of $\mathbf{0}$ [17].

**Example 4.5.** If $\dim_{\mathbf{0}} \Sigma f = 0$, then the only non-zero Lê number of $f$ is $\lambda^0_{f,z_0}(\mathbf{0})$, and we have

$$\begin{aligned}\lambda^0_{f,z_0}(\mathbf{0}) &= \operatorname{rank}_{\mathbb{Z}} H^0(\phi_{z_0}[-1]\phi_f[-1]\mathbb{Z}^\bullet_{\mathbb{C}^{n+1}}[n+1])_{\mathbf{0}} \\ &= \text{Milnor number of } f \text{ at } \mathbf{0}.\end{aligned}$$

Note that any non-zero linear form $z_0$ sufficed for this construction, since $\psi_{z_0}[-1]\phi_f[-1]\mathbb{Z}^\bullet_{\mathbb{C}^{n+1}}[n+1] = 0$.

**Remark 4.6.** Analogous to the Lê numbers $\lambda^i_{f,\mathbf{z}}(p)$, the characteristic polar multiplicities of a perverse sheaf may be expressed as intersection numbers. That is, suppose we have a perverse sheaf $\mathbf{P}^\bullet$ and a tuple of linear forms $\mathbf{z}$ such that, for all $0 \leq i \leq \dim_{\mathbf{0}} \operatorname{supp} \mathbf{P}^\bullet$, the characteristic polar multiplicities $\lambda^i_{\mathbf{P}^\bullet,\mathbf{z}}(p)$ are defined for all $p$ in a neighborhood $\mathcal{U}$ of $\mathbf{0}$. Then, there is a collection of analytic cycles $\Lambda^i_{\mathbf{P}^\bullet,\mathbf{z}}$ called the **characteristic polar cycles** of $\mathbf{P}^\bullet$ with respect to $\mathbf{z}$ such that, for all $p \in \mathcal{U}$,

$$\lambda^i_{\mathbf{P}^\bullet,\mathbf{z}}(p) = \left(\Lambda^i_{\mathbf{P}^\bullet,\mathbf{z}} \cdot V(z_0 - p_0, \cdots, \cdots, z_{i-1} - p_{i-1})\right)_p.$$

We will need this representation of the characteristic polar multiplicities in Section 5 when we will use the dynamic intersection property for proper intersections to understand $\lambda^i_{\mathbf{N}^\bullet_F,\mathbf{z}}(0)$. See [17].

**Example 4.7.** If $\dim_{\mathbf{0}} \Sigma f = 1$, then the only non-zero Lê numbers of $f$ with respect to $\mathbf{z} = (z_0, z_1)$ are $\lambda^0_{f,\mathbf{z}}(\mathbf{0})$ and $\lambda^1_{f,\mathbf{z}}(p)$ for $p \in \Sigma f$. At $\mathbf{0}$, we have

$$\lambda^1_{f,\mathbf{z}}(\mathbf{0}) = \sum_{C \subseteq \Sigma f \text{ irr.comp. at } \mathbf{0}} \mathring{\mu}_C \left(C \cdot V(z_0)\right)_{\mathbf{0}},$$

where $\mathring{\mu}_C$ denotes the generic transverse Milnor number of $f$ along $C \backslash \{0\}$. Note that we need $z_0$ such that $\dim_{\mathbf{0}} \Sigma \left(f_{|_{V(z_0)}}\right) = 0$, and any non-zero linear form suffices for $z_1$.

**Lemma 4.8.** *If $g$ is an IPA-deformation of $g_{|_{V(t)}}$, then the 0-dimensional characteristic polar multiplicity of $\mathbf{N}^\bullet_F$ with respect to $t$ is defined, and*

$$\lambda^0_{\mathbf{N}^\bullet_F,t}(\mathbf{0}) = \lambda^0_{\mathbb{Z}^\bullet_{V(g)}[n],t}(\mathbf{0}) = \left(\Gamma^1_{g,t} \cdot V(t)\right)_{\mathbf{0}}.$$

*Proof.* If $g$ is an IPA-deformation of $g_{|_{V(t)}}$ at $\mathbf{0}$, then $\dim_{\mathbf{0}} \operatorname{supp} \phi_t[-1]\mathbb{Z}^\bullet_{V(g)}[n] \leq 0$, by Proposition 2.5. By Definition 4.1, this is precisely what is needed to define $\lambda^0_{\mathbb{Z}^\bullet_{V(g)}[n],t}(\mathbf{0})$. Then, by a proper base-change, we have $\phi_t F_* \cong \hat{F}_*\phi_{t\circ F}$ where $\hat{F} : V(t\circ F) \to V(g,t)$ is the pullback of $F$ along the inclusion $V(g,t) \hookrightarrow V(g)$. But, because $F$ is a one-parameter unfolding, $t \circ F$ is a linear form on affine space and has no critical points; hence, $\phi_{t\circ F}\mathbb{Z}^\bullet_{\mathcal{U}} = 0$.

Consequently, it follows from the short exact sequence of perverse sheaves

$$0 \to \phi_t[-1]\mathbf{N}^\bullet_F \to \phi_t[-1]\mathbb{Z}^\bullet_{V(g)}[n] \to \phi_t[-1]F_*\mathbb{Z}^\bullet_{\mathcal{W}}[n] \to 0$$

that there is an equality $\lambda^0_{\mathbf{N}^\bullet_F,t}(\mathbf{0}) = \lambda^0_{\mathbb{Z}^\bullet_{V(g)}[n],t}(\mathbf{0})$, since the characteristic polar multiplicities are additive on short exact sequences.

It is then a classical result by Lê, Hamm, Teissier, and Siersma that, for sufficiently generic $t$,

$$\lambda^0_{\mathbb{Z}^\bullet_{V(g)}[n],t}(\mathbf{0}) = \left(\Gamma^1_{g,t} \cdot V(t)\right)_{\mathbf{0}},$$

the result in the generality of IPA-deformations is found in [22]. The claim then follows. □

Moreover, the iterated IPA-condition implies the higher characteristic polar multiplicities of $\mathbf{N}^\bullet$ exist as well.

**Theorem 4.9.** *Suppose that $(t,\mathbf{z}) = (t, z_1, \cdots, z_n)$ is an IPA-tuple for $g$ at $\mathbf{0}$. Then, for $0 \leq i \leq n-1$, the characteristic polar multiplicities $\lambda^i_{\mathbf{N}^\bullet_F,(t,\mathbf{z})}(\mathbf{0})$ with respect to $(t,\mathbf{z})$ are defined, and the following equalities hold:*

$$\lambda^0_{\mathbf{N}^\bullet_{f_0},\mathbf{z}}(\mathbf{0}) = \lambda^1_{\mathbf{N}^\bullet_F,(t,\mathbf{z})}(\mathbf{0}) - \lambda^0_{\mathbf{N}^\bullet_F,(t,\mathbf{z})}(\mathbf{0}),$$

*and, for $1 \leq i \leq n-2$,*

$$\lambda^i_{\mathbf{N}^\bullet_{f_0},\mathbf{z}}(\mathbf{0}) = \lambda^{i+1}_{\mathbf{N}^\bullet_F,(t,\mathbf{z})}(\mathbf{0}).$$

*Proof.* That $\lambda^0_{\mathbf{N}^\bullet_F,(t,\mathbf{z})}(\mathbf{0})$ is defined is precisely the inequality $\dim_{\mathbf{0}} \operatorname{supp} \phi_t[-1]\mathbf{N}^\bullet_F \leq 0$ concluded in Lemma 4.8 from the short exact sequence

$$0 \to \phi_t[-1]\mathbf{N}^\bullet_F \to \phi_t[-1]\mathbb{Z}^\bullet_{V(g)}[n] \to \phi_t[-1]F_*\mathbb{Z}^\bullet_{\mathcal{W}}[n] \to 0.$$

By Proposition 3.2 of [17], it remains to show that $\lambda^i_{\mathbf{N}^\bullet_F,(t,\mathbf{z})}(\mathbf{0})$ is defined for $1 \leq i \leq n-1$, i.e.,

$$\dim_{\mathbf{0}} \operatorname{supp} \phi_{z_{i-1}}[-1]\psi_{z_{i-2}}[-1]\cdots\psi_{z_1}[-1]\psi_t[-1]\mathbf{N}^\bullet_F \leq 0.$$

From the short exact sequence of perverse sheaves

$$0 \to \mathbf{N}^\bullet_F \to \mathbb{Z}^\bullet_{V(g)}[n] \to F_*\mathbb{Z}^\bullet_{\mathcal{W}}[n] \to 0,$$

it follows that $\lambda^i_{\mathbf{N}^\bullet_F,(t,\mathbf{z})}(\mathbf{0})$ is defined if $\lambda^i_{\mathbb{Z}^\bullet_{V(g)}[n],(t,\mathbf{z})}(\mathbf{0})$ is defined, by the triangle inequality for supports of perverse sheaves.

Since $(t,\mathbf{z})$ is an IPA-tuple for $g$ at $\mathbf{0}$, Proposition 2.5 gives, for $1 \leq i \leq n-1$,

$$\dim_{\mathbf{0}} \operatorname{supp} \phi_{z_i}[-1]\mathbb{Z}^\bullet_{V(g,t,z_1,\cdots,z_{i-1})}[n-i] \leq 0.$$

Thus, away from $\mathbf{0}$, each of the comparison morphisms

$$\mathbb{Z}^\bullet_{V(g,t,z_1,\cdots,z_{i-1},z_i)}[n-i-1] \xrightarrow{\sim} \psi_{z_i}[-1]\mathbb{Z}^\bullet_{V(g,t,z_1,\cdots,z_{i-1})}[n-i]$$

is an isomorphism for $1 \leq i \leq n-1$. Consequently,

$$\dim_{\mathbf{0}} \operatorname{supp} \phi_{z_i}[-1]\mathbb{Z}^{\bullet}_{V(g,t,z_1,\cdots,z_{i-1})}[n-i] \leq 0$$

implies

$$\dim_{\mathbf{0}} \operatorname{supp} \phi_{z_i}[-1]\psi_{z_{i-1}}[-1]\cdots\psi_{z_1}[-1]\psi_t[-1]\mathbb{Z}^{\bullet}_{V(g)}[n] \leq 0,$$

and the claim follows.

□

## 5. The Main Result

We wish to express the Lê numbers of $g_0$ entirely in terms of data from the Lê numbers of $g_{t_0}$ and the characteristic polar multiplicities of both $\mathbf{N}^{\bullet}_{f_0}$ and $\mathbf{N}^{\bullet}_{f_{t_0}}$, for $t_0$ small and nonzero. The starting point is Proposition 2.10:

$$\lambda^0_{g_0,\mathbf{z}}(\mathbf{0}) = \left(\Gamma^1_{g,t} \cdot V(t)\right)_{\mathbf{0}} + \lambda^1_{g,(t,\mathbf{z})}(\mathbf{0})$$
$$\lambda^i_{g_{t_0},\mathbf{z}}(\mathbf{0}) = \lambda^{i+1}_{g,(t,\mathbf{z})}(\mathbf{0}),$$

where $(t,\mathbf{z}) = (t, z_1, \cdots, z_n)$ is an IPA-tuple for $g$ at $\mathbf{0}$. From Lemma 4.8, we have $\left(\Gamma^1_{g,t} \cdot V(t)\right)_{\mathbf{0}} = \lambda^0_{\mathbf{N}^{\bullet}_F,(t,\mathbf{z})}(\mathbf{0})$; we now have all our relevant data in terms of Lê numbers and characteristic polar multiplicities of $\mathbf{N}^{\bullet}_F$. The goal is then to decompose this data into only numerical invariants which refer to the $t = 0$ and $t \neq 0$ slices of $V(g)$.

So, in order to realize this goal, the next step is to decompose $\lambda^0_{\mathbf{N}^{\bullet}_F,(t,\mathbf{z})}(\mathbf{0})$ and $\lambda^i_{g,(t,\mathbf{z})}(\mathbf{0})$ for $i \geq 1$.

The 1-dimensional Lê number $\lambda^1_{g,(t,\mathbf{z})}(\mathbf{0})$ is the easiest; by the dynamic intersection property for proper intersections,

$$\begin{aligned}\lambda^1_{g,(t,\mathbf{z})}(\mathbf{0}) &= \left(\Lambda^1_{g,(t,\mathbf{z})} \cdot V(t)\right)_{\mathbf{0}} \\ &= \sum_{p \in B_\epsilon \cap V(t-t_0)} \left(\Lambda^1_{g,(t,\mathbf{z})} \cdot V(t-t_0)\right)_p \\ &= \sum_{p \in B_\epsilon \cap V(t-t_0)} \lambda^0_{g_{t_0},\mathbf{z}}(p).\end{aligned}$$

The approach for $\lambda^i_{g_0,\mathbf{z}}(\mathbf{0})$ for $i \geq 1$ is similar; we will use the fact that $g$ is an IPA-deformation of $g_0$ to "move" around the origin in the $V(t)$ slice, and then use the dynamic intersection property.

**Proposition 5.1.** *If $(t,\mathbf{z}) = (t, z_1, \cdots, z_i)$ is an IPA-tuple for $g$ at $\mathbf{0}$ for $i \geq 1$, the following equality of intersection numbers holds:*

$$\lambda^i_{g_0,\mathbf{z}}(\mathbf{0}) = \sum_{q \in B_\epsilon \cap V(t-t_0, z_1, z_2, \cdots, z_i)} \lambda^i_{g_{t_0},\mathbf{z}}(q)$$

*where $0 < |t_0| \ll |\xi| \ll \epsilon \ll 1$*

*Proof.* First, recall that $\lambda^i_{g_0,\mathbf{z}}(\mathbf{0}) = \left(\Lambda^i_{g_0,\mathbf{z}} \cdot V(z_1,\cdots,z_i)\right)_{\mathbf{0}}$, where $\Lambda^i_{g_0,\mathbf{z}}$ is the $i$-dimensional Lê cycle of $g_0$ with respect to $\mathbf{z}$ (see the appendix, as well as [15]). For $i \geq 1$, we have

$$\Lambda^i_{g_0,\mathbf{z}} = \Lambda^{i+1}_{g,(t,\mathbf{z})} \cdot V(t),$$

so, by the dynamic intersection property,

$$\begin{aligned}\lambda^i_{g_0,\mathbf{z}}(\mathbf{0}) &= \left(\Lambda^{i+1}_{g,(t,\mathbf{z})} \cdot V(t, z_1, \cdots, z_i)\right)_{\mathbf{0}} \\ &= \sum_{q \in B_\epsilon \cap V(t-t_0, z_1, z_2, \cdots, z_i)} \left(\Lambda^{i+1}_{g,(t,\mathbf{z})} \cdot V(t-t_0, z_1, z_2, \cdots, z_i)\right)_q \\ &= \sum_{q \in B_\epsilon \cap V(t-t_0, z_1, z_2, \cdots, z_i)} \left(\Lambda^{i}_{g_{t_0},\mathbf{z}} \cdot V(z_1, z_2, \cdots, z_i)\right)_q \\ &= \sum_{q \in B_\epsilon \cap V(t-t_0, z_1-\xi, z_2, \cdots, z_i)} \lambda^i_{g_{t_0},\mathbf{z}}(\mathbf{0}),\end{aligned}$$

where the second equality follows from the equality of cycles $\Lambda^{i+1}_{g,\mathbf{z}} \cdot V(t-t_0) = \Lambda^i_{g_{t_0},\mathbf{z}}$. □

We can now state and prove our main result.

**Theorem 5.2.** *Suppose that $F : (\mathcal{W}, \{0\} \times S) \to (\mathcal{U}, \mathbf{0})$ is a one-parameter unfolding with parameter $t$, with $\operatorname{im} F = V(g)$ for some $g \in \mathcal{O}_{\mathcal{U},\mathbf{0}}$. Suppose further that $\mathbf{z} = (z_1, \cdots, z_n)$ is chosen such that $\mathbf{z}$ is an IPA-tuple for $g_0 = g_{|_{V(t)}}$ at $\mathbf{0}$. Then, the following formulas hold for the Lê numbers of $g_0$ with respect to $\mathbf{z}$ at $\mathbf{0}$: for $0 < |t_0| \ll \epsilon \ll 1$,*

$$\lambda^0_{g_0,\mathbf{z}}(\mathbf{0}) = -\lambda^0_{\mathbf{N}^\bullet_{f_0},\mathbf{z}}(\mathbf{0}) + \sum_{p \in B_\epsilon \cap V(t-t_0)} \left(\lambda^0_{g_{t_0},\mathbf{z}}(p) + \lambda^0_{\mathbf{N}^\bullet_{f_{t_0}},\mathbf{z}}(p)\right),$$

*and, for $1 \leq i \leq n-2$,*

$$\lambda^i_{g_0,\mathbf{z}}(\mathbf{0}) = \sum_{q \in B_\epsilon \cap V(t-t_0, z_1, z_2, \cdots, z_i)} \lambda^i_{g_{t_0},\mathbf{z}}(q).$$

*In particular, the following relationship holds for $0 \leq i \leq n-2$:*

$$\lambda^i_{g_0,\mathbf{z}}(\mathbf{0}) + \lambda^i_{\mathbf{N}^\bullet_{f_0},\mathbf{z}}(\mathbf{0}) = \sum_{p \in B_\epsilon \cap V(t-t_0, z_1, z_2, \cdots, z_i)} \left(\lambda^i_{g_{t_0},\mathbf{z}}(p) + \lambda^i_{\mathbf{N}^\bullet_{f_{t_0}},\mathbf{z}}(p)\right)$$

*Proof.* By Proposition 2.10 and Proposition 5.1, it suffices to prove

$$\lambda^0_{\mathbf{N}^\bullet_F,(t,\mathbf{z})}(\mathbf{0}) = -\lambda^0_{\mathbf{N}^\bullet_{f_0},\mathbf{z}}(\mathbf{0}) + \sum_{p \in B_\epsilon \cap V(t-t_0)} \lambda^0_{\mathbf{N}^\bullet_{f_{t_0}},\mathbf{z}}(p).$$

Since $(t, \mathbf{z})$ is an IPA-tuple for $g$ at $\mathbf{0}$, Theorem 4.9 yields

$$\lambda^0_{\mathbf{N}^\bullet_{f_0},\mathbf{z}}(\mathbf{0}) = \lambda^1_{\mathbf{N}^\bullet_F,(t,\mathbf{z})}(\mathbf{0}) - \lambda^0_{\mathbf{N}^\bullet_F,(t,\mathbf{z})}(\mathbf{0}),$$

where $\mathbf{N}^\bullet_{f_0} \cong \mathbf{N}^\bullet_{F|_{V(t)}}[-1]$ (cf. Remark 3.7).

The main claim then follows by the dynamic intersection property for proper intersections applied to $\Lambda^1_{\mathbf{N}^\bullet,(t,\mathbf{z})}$ (see Remark 4.6):

$$\begin{aligned}\lambda^1_{\mathbf{N}^\bullet_F,(t,\mathbf{z})}(\mathbf{0}) &= \left(\Lambda^1_{\mathbf{N}^\bullet_F,(t,\mathbf{z})}\cdot V(t)\right)_{\mathbf{0}} \\ &= \sum_{p\in B_\epsilon\cap V(t-t_0)} \left(\Lambda^1_{\mathbf{N}^\bullet_F,(t,\mathbf{z})}\cdot V(t-t_0)\right)_p \\ &= \sum_{p\in B_\epsilon\cap V(t-t_0)} \lambda^0_{\mathbf{N}^\bullet_{f_{t_0}},\mathbf{z}}(p),\end{aligned}$$

for $0<|t_0|\ll\epsilon\ll 1$.

Finally, we examine the relationship

$$\lambda^i_{g_0,\mathbf{z}}(\mathbf{0}) + \lambda^i_{\mathbf{N}^\bullet_{f_0},\mathbf{z}}(\mathbf{0}) = \sum_{p\in B_\epsilon\cap V(t-t_0,z_1,z_2,\cdots,z_i)} \left(\lambda^i_{g_{t_0},\mathbf{z}}(p) + \lambda^i_{\mathbf{N}^\bullet_{f_{t_0}},\mathbf{z}}(p)\right).$$

For $i=0$, this follows by a trivial rearrangement of the terms in our expression for $\lambda^0_{g_0,\mathbf{z}}(\mathbf{0})$. For $i\geq 1$, this is just Proposition 5.1 combined with Theorem 4.9 and the dynamic intersection property on $\lambda^i_{\mathbf{N}^\bullet_F,(t,\mathbf{z})}(\mathbf{0})$, as in Proposition 5.1 for $\lambda^i_{g,(t,\mathbf{z})}(\mathbf{0})$. □

**Example 5.3.** We wish to examine Theorem 5.2 in the context of Milnor's double point formula, where $F:(\mathbb{C}^2,\{0\}\times S)\to(\mathbb{C}^3,\mathbf{0})$ parameterizes a deformation of the curve $V(g_0)$ into a curve $V(g_{t_0})$ with only double-point singularities. In this case, $\dim_{\mathbf{0}}\Sigma g_0=0$, so the only non-zero Lê number of $g_0$ is $\lambda^0_{g_0,z}(\mathbf{0})$, where $z$ is any non-zero linear form on $\mathbb{C}^2$, and $\lambda^0_{g_0,z}(\mathbf{0})=\mu_{\mathbf{0}}(g_0)$.

It is then an easy exercise to see that $\lambda^0_{\mathbf{N}^\bullet_{f_{t_0}},z}(p)=|F^{-1}(p)|-1$ for $t_0$ small (and possibly zero) and $p\in D$.

All together, this gives, by Theorem 5.2

$$\begin{aligned}\mu_{\mathbf{0}}(g_0) &= -(r-1) + \sum_{p\in B_\epsilon\cap V(t-t_0)} \left(\mu_p(g_{t_0}) + |F^{-1}(p)|-1\right) \\ &= 2\delta - r + 1,\end{aligned}$$

as there are $\delta$ double-points in the deformed curve $V(g_{t_0})$. We have thus recovered Milnor's original double-point formula for the Milnor number of a plane curve singularity.

## 6. Future Directions

The results obtained in Theorem 5.2 point to several possible future directions for research.

### 6.1. Conservation of Number Property.

The relationship in Theorem 5.2,

$$\lambda^i_{g_0,\mathbf{z}}(\mathbf{0}) + \lambda^i_{\mathbf{N}^\bullet_{f_0},\mathbf{z}}(\mathbf{0}) = \sum_{p\in B_\epsilon\cap V(t-t_0,z_1,z_2,\cdots,z_i)} \left(\lambda^i_{g_{t_0},\mathbf{z}}(p) + \lambda^i_{\mathbf{N}^\bullet_{f_{t_0}},\mathbf{z}}(p)\right),$$

suggests a sort of "conservation of number" property for the sum of Lê numbers of $g_{t_0}$ and characteristic polar multiplicities of $\mathbf{N}^\bullet_{f_{t_0}}$ for $t_0$ small (and possibly zero).

We are very interested in exploring this relationship and its possible implications for understanding the vanishing cycles in the derived category on $V(g)$.

6.2. **Parameterized Local Complete Intersections.** In Section 3 (and further, in [8]), we defined $\mathbf{N}_F^\bullet$ to be the perverse sheaf that fits into the short exact sequence

$$0 \to \mathbf{N}_F^\bullet \to \mathbb{Z}_{V(g)}^\bullet[n] \to F_*\mathbb{Z}_{\mathcal{W}}^\bullet[n] \to 0$$

of perverse sheaves on $V(g)$. If one instead parameterizes an $n$-dimensional local complete intersection (LCI) $X$ by a finite, generically one-to-one morphism $F$, there is an analogous short exact sequence of perverse sheaves on $X$:

$$0 \to \mathbf{N}_F^\bullet \to \mathbb{Z}_X^\bullet[n] \to F_*\mathbb{Z}_{\mathcal{W}}^\bullet[n] \to 0$$

since the shifted constant sheaf $\mathbb{Z}_X^\bullet[n]$ is perverse for an LCI. The resulting multiple-point complex $\mathbf{N}_F^\bullet$ still satisfies all the properties of Proposition 3.6. One can also consider IPA-deformations of LCI's, with some additional technical assumptions.

In this setting, can we find deformation-type formulas for the characteristic polar multiplicities of $\mathbf{N}_F^\bullet$ and the vanishing cycles of the functions defining $X$, in a manner analogous to Theorem 5.2?

## 7. Appendix: Lê Numbers and Lê Cycles

The Lê numbers of a function with a non-isolated critical locus are the fundamental invariants we consider in this paper. First defined by Massey in [19] and [20], these numbers generalize the Milnor number of a function with an isolated critical point.

The Lê cycles and numbers of $g$ are classically defined with respect to a **prepolar-tuple** of linear forms $\mathbf{z} = (z_0, \cdots, z_n)$; loosely, these are linear forms that transversely intersect all strata of a good stratification of $V(g)$ near $\mathbf{0}$ (see, for example, Definition 1.26 of [15]). The purpose of Proposition 2.10 in Section 2 is to replace the assumption of prepolar-tuples with IPA tuples.

**Definition 7.1.** *The $k$-**dimensional relative polar variety of** $g$ **with respect to** $\mathbf{z}$, at the origin, denoted $\Gamma_{g,\mathbf{z}}^k$, consists of those components of the analytic cycle $V\left(\frac{\partial g}{\partial z_k}, \cdots, \frac{\partial g}{\partial z_n}\right)$ at the origin which* are not *contained in $\Sigma g$.*

**Definition 7.2.** *The $k$-**dimensional Lê cycle of** $g$ **with respect to** $\mathbf{z}$, at the origin, denoted $\Lambda_{g,\mathbf{z}}^k$, consists of those components of the analytic cycle $\Gamma_{g,\mathbf{z}}^{k+1} \cdot V\left(\frac{\partial g}{\partial z_k}\right)$ which* are *contained in $\Sigma g$.*

**Definition 7.3.** *The $k$-**dimensional Lê number of** $g$ **at** $p = (p_0, \cdots, p_n)$ **with respect to** $\mathbf{z}$, denoted $\lambda_{g,\mathbf{z}}^k(p)$, is equal to the intersection number*

$$\left(\Lambda_{g,\mathbf{z}}^k \cdot V(z_0 - p_0, \cdots, z_{k-1} - p_{k-1})\right)_p,$$

*provided this intersection is purely zero-dimensional at $p$.*

**Example 7.4.** When $g$ has an isolated critical point at the origin, the only non-zero Lê number of $g$ is $\lambda^0_{g,\mathbf{z}}(\mathbf{0})$. In this case, we have:

$$\begin{aligned}\lambda^0_{g,\mathbf{z}}(\mathbf{0}) &= \left(\Lambda^0_{g,\mathbf{z}} \cdot \mathcal{U}\right)_{\mathbf{0}} \\ &= V\left(\frac{\partial g}{\partial z_0}, \cdots, \frac{\partial g}{\partial z_n}\right)_{\mathbf{0}},\end{aligned}$$

i.e., the 0-dimensonal Lê number of $g$ is just the multiplicity of the Jacobian scheme. In the case of an isolated critical point, this is the Milnor number of $g$ at $\mathbf{0}$.

**Example 7.5.** Suppose now that $\dim_{\mathbf{0}} \Sigma g = 1$. Then, the only non-zero Lê numbers of $g$ are $\lambda^0_{g,\mathbf{z}}(\mathbf{0})$ and $\lambda^1_{g,\mathbf{z}}(p)$ for $p \in \Sigma g$.

At $\mathbf{0}$, we have

$$\begin{aligned}\lambda^1_{g,\mathbf{z}}(\mathbf{0}) &= \left(\Lambda^1_{g,\mathbf{z}} \cdot V(z_0)\right)_{\mathbf{0}} \\ &= \left(V\left(\frac{\partial g}{\partial z_1}, \cdots, \frac{\partial g}{\partial z_n}\right) \cdot V(z_0)\right)_{\mathbf{0}} \\ &= \sum_{q \in B_\epsilon \cap V(z_0 - q_0) \cap \Sigma g} \left(V\left(\frac{\partial g}{\partial z_1}, \cdots, \frac{\partial g}{\partial z_n}\right) \cdot V(z_0 - q_0)\right)_q \\ &= \sum_{q \in B_\epsilon \cap V(z_0 - q_0) \cap \Sigma g} \mu_q\left(g_{|_{V(z_0 - q_0)}}\right)\end{aligned}$$

where the second to last line is the dynamic intersection property for proper intersections.

After rearranging the terms in the last line, we find

$$\lambda^1_{g,\mathbf{z}}(\mathbf{0}) = \sum_{C \subseteq \Sigma g \text{ irred. comp.}} \overset{\circ}{\mu}_C \left(C \cdot V(z_0)\right)_{\mathbf{0}},$$

where the sum is indexed over the collection of irreducible components of $\Sigma g$ at the origin, and $\overset{\circ}{\mu}_C$ denotes the generic transversal Milnor number of $g$ along $C$.

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