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Perverse Results on Milnor Fibers inside Parameterized Hypersurfaces

by

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Abstract

We discuss some results for the cohomology of Milnor fibers inside parameterized hypersurfaces which follow quickly from results in the category of perverse sheaves. In particular, we define a new perverse sheaf called the multiple-point complex of the parameterization, which naturally arises when investigating how the multiple-point set influences the topology of the Milnor fiber. We also discuss applications to stable unfoldings of finite maps with isolated instabilities.

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§1. Introduction

Throughout this paper, $\mathcal{U}$ will denote an open neighborhood of the origin in $\mathbb{C}^{n+1}$, $\mathcal{W}$ will denote an open subset of $\mathbb{C}^n$ (or an open subset of a finite number of disjoint copies of $\mathbb{C}^n$), S will denote a finite set of r points $\{p_1, \dots, p_r\}$ in $\mathcal{W}$, and $F : (\mathcal{W}, S) \to (\mathcal{U}, \mathbf{0})$ will denote a finite, complex analytic map which is generically one-to-one such that $F^{-1}(\mathbf{0}) = S$.

We are interested in the germ of the image of F at the origin. By shrinking $\mathcal{W}$ and $\mathcal{U}$ if necessary, we can, and do, assume that $\mathcal{W}$ consists of r disjoint, connected, open sets, $\mathcal{W}_1, \dots, \mathcal{W}_r$ and, for $1 \leq i \leq r$, $F^{-1}(\mathbf{0}) \cap \mathcal{W}_i = \{p_i\}$. The case where $r = 1$ is usually referred to as the *mono-germ* case, and the case where $r > 1$ as the *multi-germ* case.

In our setting, the Finite Mapping Theorem [GR] tells us that the image of F is a complex analytic space of dimension n, i.e., is a hypersurface $X := V(g)$ in

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$\mathcal{U}$, for some complex analytic $g : \mathcal{U} \to \mathbb{C}$. We will continue to use F to denote the surjection $F : \mathcal{W} \to X$.

The multi-germ case allows us to deal with hypersurfaces X which have more than one irreducible component at the origin and so, in particular, we can obtain results in the case where X is a union of hyperplanes containing the origin.

Another way of thinking of $F : \mathcal{W} \to X$ is as a finite resolution of singularities. In particular, F is a small resolution in the sense of Goresky-MacPherson and, consequently, the shifted constant sheaf $\mathbb{Z}^\bullet_{\mathcal{W}}[n]$ on $\mathcal{W}$ pushes forward by F to the intersection cohomology complex $\mathbf{I}^\bullet_X$ on X (see [GM]).

The stalk cohomology of $\mathbf{I}^\bullet_X$ is trivial to describe. For each $x \in X$, let $m(x)$ denote the number of points in the inverse image of F (**without** multiplicity), i.e., $m(x) := \left|F^{-1}(x)\right|$. Note that $m(\mathbf{0}) = r$. Then the stalk cohomology of $\mathbf{I}^\bullet_X$ is given by, for all $x \in X$,

$$H^k(\mathbf{I}^\bullet_X)_x \cong \begin{cases} \mathbb{Z}^{m(x)}, & \text{if } k = -n; \\ 0, & \text{otherwise.} \end{cases}$$

In this paper, we will use general properties and results from the derived category and the Abelian category of perverse sheaves to investigate the cohomology of Milnor fibers of complex analytic functions $h : X \to \mathbb{C}$. We outline these results below.

For $k \geq 1$, let $X_k := \{x \in X \mid m(x) = k\} = m^{-1}(k)$, and let

$$D := \overline{\bigcup_{k\geq 2} X_k},$$

which is the closure of the image of the double-point (or multiple-point) set with its reduced structure. Note that, since we are taking the closure, D may contain points of X_1. Later, we shall show that D is contained in the singular set ΣX of X.

Suppose now that we have a complex analytic function $h : (X, \mathbf{0}) \to (\mathbb{C}, 0)$.

We are interested in results on the Milnor fiber, $M_{h,\mathbf{0}}$ of h at $\mathbf{0}$. We remind the reader that, in this context in which the domain of h is allowed to be singular, a Milnor fibration still exists by the result of Lê in [L1], and the Milnor fiber at a point $x \in V(h)$, is given by

$$M_{h,x} \ = \ B^\circ_\epsilon(x) \cap X \cap h^{-1}(a),$$

where $B^\circ_\epsilon(x)$ is an open ball of radius ϵ, centered at x, and $0 < |a| \ll \epsilon \ll 1$ (and, technically, the intersection with X is redundant, but we wish to emphasize that this Milnor fiber lives in X). We also care about the real link, $K_{_{X,x}}$, of X at $x \in X$ [Mi], which is given by

$$K_{_{X,x}} \ := \ \partial B_\epsilon(x) \cap X = S_\epsilon(x) \cap X,$$

where, again, $0 < \epsilon \ll 1$.

We will need to consider the Milnor fiber of $h \circ F$ at each of the p_i and the Milnor fiber of h restricted to the X_k's, which are equal to the intersections $X_k \cap M_{h,\mathbf{0}}$.

As X itself may be singular, it is important for us to say what notion we will use for a "critical point" of h. We use the Milnor fiber to define:

Definition 1.1. The *topological/cohomological critical locus of* h, is

$$\Sigma_{_{\rm top}} h := \{x \in V(h) \mid M_{h,x} \text{ does not have the integral cohomology of a point}\}.$$

Remark 1.2. Suppose, for instance, that F is a stable unfolding of a finite map f, and that h is the projection onto one of the unfolding parameters. Then a point $x \in V(h)$ is a point in the image of f. If f is stable at x, then h is locally a topologically trivial fibration in a neighborhood of x; consequently, the Milnor fiber is contractible, and $x \notin \Sigma_{_{\rm top}} h$.

Thus, $\Sigma_{_{\rm top}} h$ is contained in the unstable locus of f.

Now, F induces a finite map $\widetilde{F}$ from the union of the Milnor fibers $M_{h\circ F,p_i}$ in the domain of F to the Milnor fiber $M_{h,\mathbf{0}}$, which can be stratified in the sense of Goresky and MacPherson [GM] in such a way that the closure of each $X_k \cap M_{h,\mathbf{0}}$ is a union of strata. From this, via a Riemann-Hurwitz-type argument, it is not difficult to show that the Euler characteristics are related by

$$\sum_{1\leq i\leq r} \chi(M_{h\circ F,p_i}) = \sum_{k\geq 1} k \cdot \chi(X_k \cap M_{h,\mathbf{0}}) = \chi(M_{h,\mathbf{0}}) + \sum_{k\geq 2} (k-1)\cdot \chi(X_k \cap M_{h,\mathbf{0}}).$$

Or, rearranging and writing $\widetilde{\chi}$ for the Euler characteristic of the reduced cohomology (in order to focus on vanishing cohomology), we obtain

$$(\star) \qquad \widetilde{\chi}(M_{h,\mathbf{0}}) = r - 1 + \sum_i \widetilde{\chi}(M_{h\circ F,p_i}) - \sum_{k\geq 2}(k-1)\cdot \chi(X_k \cap M_{h,\mathbf{0}}).$$

Equation ($\star$) is particularly interesting in the case where the reduced cohomology of $M_{h,\mathbf{0}}$ is concentrated in a single degree and the reduced cohomology of $M_{h\circ F,p_i}$ is zero.

In this paper, we show:

1. If $s := \dim_{\mathbf{0}} \Sigma_{_{\mathrm{top}}} h$, then the reduced cohomology of $M_{h,\mathbf{0}}$ can be non-zero only in degrees k where $n-1-s \leq k \leq n-1$, and is free Abelian in degree $n-1-s$. In particular, if $\mathbf{0}$ is an isolated point in $\Sigma_{_{\mathrm{top}}} h$, then $M_{h,\mathbf{0}}$ has the cohomology of a bouquet of $(n-1)$-spheres.
2. As discussed above, there is a relationship between the reduced Euler characteristics of the Milnor fiber $M_{h,\mathbf{0}}$, the Milnor fibers $M_{h\circ F,p_i}$, and the Milnor fibers of the X_k's, given by
$$\widetilde{\chi}(M_{h,\mathbf{0}}) = r - 1 + \sum_i \widetilde{\chi}(M_{h\circ F,p_i}) - \sum_{k\geq 2}(k-1)\cdot\chi(X_k \cap M_{h,\mathbf{0}}).$$
3. There is a perverse sheaf $\mathbf{N}^\bullet$, supported on D, with the properties that:
    - for all $x \in D$, the stalk cohomology of $\mathbf{N}^\bullet$ at x is (possibly) non-zero in a single degree, degree $-n+1$, where it is isomorphic to $\mathbb{Z}^{m(x)-1}$;
    - With some special assumptions on h, there is a long exact sequence, relating the Milnor fiber of h, the Milnor fibers of $h \circ F$, and the hypercohomology of the Milnor fiber of h restricted to D with coefficients in $\mathbf{N}^\bullet[-n+1]$, given by
    $$\cdots \to \widetilde{\mathbb{H}}^{j-1}(D\cap M_{h,\mathbf{0}};\mathbf{N}^\bullet[-n+1]) \to \widetilde{H}^j(M_{h,\mathbf{0}};\mathbb{Z}) \to \bigoplus_i \widetilde{H}^j(M_{h\circ F,p_i};\mathbb{Z}) \to \widetilde{\mathbb{H}}^j(D\cap M_{h,\mathbf{0}};\mathbf{N}^\bullet[-n+1]) \to \cdots ,$$
    where the reduced cohomology with coefficients in $\mathbf{N}^\bullet[-n+1]$ has the special meaning of reducing the rank by $r-1$ in degree zero and having no effect in other degrees.

      This long exact sequence is compatible with the Milnor monodromy automorphisms in each degree.
    - In particular, if $S\cap\Sigma(h\circ F) = \emptyset$, then the reduced cohomology $\widetilde{H}^j(M_{h,\mathbf{0}};\mathbb{Z})$ is isomorphic to the reduced hypercohomology
    $$\widetilde{\mathbb{H}}^{j-1}\left(D\cap M_{h,\mathbf{0}};\mathbf{N}^\bullet[-n+1]\right),$$
    by an isomorphism which commutes with the respective Milnor monodromies.

4. Suppose that $\mathbf{0}$ is an isolated point in $\Sigma_{\text{top}} h$ and that $S \cap \Sigma(h \circ F) = \emptyset$. Then,

$$\widetilde{H}^{n-1}(M_{h,\mathbf{0}};\mathbb{Z}) \;\cong\; \mathbb{Z}^{\omega} \;\cong\; \widetilde{\mathbb{H}}^{n-2}\left(D \cap M_{h,\mathbf{0}};\mathbf{N}^\bullet[-n+1]\right),$$

where $\omega := (-1)^{n-1}\left[(r-1) - \sum_{k\geq 2}(k-1)\chi(X_k \cap M_{h,\mathbf{0}})\right]$.

5. Suppose that $n=2$ and that F is a one-parameter unfolding of a parameterization f of a plane curve singularity with r irreducible components at the origin. Let t be the unfolding parameter and suppose that the only singularities of $M_{t,\mathbf{0}}$ are nodes, and that there are δ of them. Recall that $X = V(g)$, and let $g_0 := g_{|_{V(t)}}$. Then, we recover the classical formula for the Milnor number of g_0, as given in Theorem 10.5 of [Mi]:

$$\mu_{\mathbf{0}}\left(g_0\right) = 2\delta - r + 1.$$

§2. A Standard Vanishing Result

Before we state the only result of this section, we need to establish a convention for a degenerate case: the reduced cohomology of the empty set.

Convention: We define the reduced cohomology of the empty set, $\widetilde{H}^k(\emptyset;\mathbb{Z})$, to be zero in all degrees other than degree -1, and we define $\widetilde{H}^{-1}(\emptyset;\mathbb{Z}) = \mathbb{Z}$.

We do this so that the stalk cohomology at p of the vanishing cycles of the constant sheaf along a complex analytic function $f:(E,p)\to(\mathbb{C},0)$ always yields the reduced cohomology of the Milnor fiber of f at p, even in the case where f is identically zero on E. This is true because, if B is the intersection with E of a small open ball around p in some ambient affine space (after embedding), then

$$H^k(\phi_f\mathbb{Z}^\bullet_E)_p \cong H^{k+1}(B, M_{f,p};\mathbb{Z}).$$

One then looks at the long exact sequence of the pair $(B, M_{f,p})$, paying special attention to the case where $M_{f,p}=\emptyset$, i.e., the case where f is identically zero.

The following result is, by now, a well-known consequence of the general theory of perverse sheaves and vanishing cycles. Nonetheless, we give a quick proof.

Proposition 2.1. *If $s := \dim_{\mathbf{0}}\Sigma_{\text{top}} h$, then the reduced integral cohomology of $M_{h,\mathbf{0}}$ can be non-zero only in degrees k where $n-1-s \leq k \leq n-1$, and is free Abelian in degree $n-1-s$.*

In particular, if $\mathbf{0}$ is an isolated point in $\Sigma_{\text{top}} h$, then $M_{h,\mathbf{0}}$ has the integral cohomology of a bouquet of $(n-1)$-spheres.

Proof. By the result of Lê in [L2], if X is a purely n-dimensional local complete intersection, and S is a d-dimensional stratum in a Whitney stratification of X, then the complex link of S has the homotopy-type of a finite bouquet of $(n-1-d)$-spheres.

The cohomological implication is that the constant sheaves $\mathbb{Z}^\bullet_X[n]$ and $(\mathbb{Z}/p\mathbb{Z})^\bullet_X[n]$, for p prime, are perverse sheaves. Consequently, the shifted vanishing cycles

$$\phi_h[-1]\mathbb{Z}^\bullet_X[n] \quad \text{and} \quad \phi_h[-1]\,(\mathbb{Z}/p\mathbb{Z})^\bullet_X\,[n]$$

are also perverse, and have support contained in the closure $\overline{\Sigma_{\text{top}} h}$.

Hence, these vanishing cycles have possibly non-zero stalk cohomology in degrees k such that $-s \leq k \leq 0$. This means that the reduced cohomology of the Milnor fiber of h at $\mathbf{0}$, with coefficients in $\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z}$, is possibly non-zero in degrees $n-1-s$ through $n-1$. This proves the result, except for the free Abelian claim in degree $n-1-s$.

However, that is the point of the $\mathbb{Z}/p\mathbb{Z}$ discussion. As $\widetilde{H}^{n-2-s}(M_{h,\mathbf{0}}; \mathbb{Z}/p\mathbb{Z}) = 0$, for all p, the Universal Coefficient Theorem tells us that $\widetilde{H}^{n-1-s}(M_{h,\mathbf{0}}; \mathbb{Z})$ has no torsion. $\square$

For a stable unfolding F with an isolated instability and projection h onto an unfolding parameter, the result above is a cohomological generalization of the result of Mond in [Mo].

Definition 2.2. If $\mathbf{0}$ is an isolated point in $\Sigma_{\text{top}} h$, then we define the *Milnor number of h at* $\mathbf{0}$, $\mu_{\mathbf{0}}(h)$, to be the rank of $\widetilde{H}^{n-1}(M_{h,\mathbf{0}}; \mathbb{Z})$.

§3. The Push-forward of the Constant Sheaf

General references for the derived category techniques in this section and the next are [KS], [D], and [GM]. As we are always considering the derived category, we follow the usual practice of omitting the "R"s in front of right derived functors.

We made the following definition in the introduction.

Definition 3.1. Let $\mathbf{I}^\bullet_X$ denote the (derived) push-forward of the constant sheaf $\mathbb{Z}^\bullet_{\mathcal{W}}[n]$; that is, $\mathbf{I}^\bullet_X := F_*\mathbb{Z}^\bullet_{\mathcal{W}}[n]$.

In the notation $\mathbf{I}^\bullet_X$, we will justify subscripting by X, rather than by F, below.

Proposition 3.2. *The complex* $\mathbf{I}^\bullet_X = F_*\mathbb{Z}^\bullet_{\mathcal{W}}[n]$ *has the following properties:*

1. $\mathbf{I}^\bullet_X$ *is the intersection cohomology complex with the constant* $\mathbb{Z}$ *local system.*
2. *The stalk cohomology of* $\mathbf{I}^\bullet_X$ *is given by for all* $x \in X$,
$$H^k(\mathbf{I}^\bullet_X)_x \cong \begin{cases} \mathbb{Z}^{m(x)}, & \text{if } k = -n; \\ 0, & \text{otherwise.} \end{cases}$$
3. *The complex* $\mathbf{I}^\bullet_X$ *is self-Verdier dual, i.e.,*
$$\mathcal{D}\mathbf{I}^\bullet_X \cong \mathbf{I}^\bullet_X.$$
4. *Suppose* $x \in X$, *and* j_x *denotes the inclusion of* x *into* X. *Then,*
$$j_x^!\mathbf{I}^\bullet_X \cong \mathcal{D}j_x^*\mathcal{D}\mathbf{I}^\bullet_X \cong \mathcal{D}j_x^*\mathbf{I}^\bullet_X$$
and so the costalk cohomology is given by
$$H^k(j_x^!\mathbf{I}^\bullet_X) \cong \begin{cases} \mathbb{Z}^{m(x)}, & \text{if } k = n; \\ 0, & \text{otherwise.} \end{cases}$$
5. *There is a canonical surjection of perverse sheaves* $\mathbb{Z}^\bullet_X[n] \xrightarrow{c} \mathbf{I}^\bullet_X$, *which induces the diagonal map on the stalk cohomology.*

Proof.

1. As $\mathbb{Z}^\bullet_{\mathcal{W}}[n]$ is the intersection cohomology complex on $\mathcal{W}$, with constant coefficients, and $\mathbf{I}^\bullet_X$ is its push-forward by a finite map, the support and cosupport conditions trivially push forward, and $\mathbf{I}^\bullet_X$ is an intersection cohomology complex on X.
A priori, $\mathbf{I}^\bullet_X$ could have "twisted" coefficients in a non-trivial local system on the regular part, X_{reg}, of X. However, as we are assuming that F is generically one-to-one, F induces a homeomorphism when restricted to a map from a generic subset of $\mathcal{W}$ to a generic subset of X_{reg}. Thus, on a generic subset of X_{reg}, the complex $\mathbf{I}^\bullet_X$ restricts to the shifted constant sheaf, and so $\mathbf{I}^\bullet_X$ is the intersection cohomology complex with the constant local system.
Alternatively, F is a small resolution of X, and so the push-forward of the shifted constant sheaf yields intersection cohomology. [GM]

2. The formula for the stalk cohomology of $\mathbf{I}^\bullet_X$ is immediate since $\mathbf{I}^\bullet_X := F_*\mathbb{Z}^\bullet_{\mathcal{W}}[n]$.

3. With a field for the base ring, the self-duality of $\mathbf{I}^\bullet_X$ would follow from its being the intersection cohomology complex. However, since we are using $\mathbb{Z}$ as our base ring, we use that
$$\mathcal{D}\mathbf{I}^\bullet_X \cong \mathcal{D}F_*\mathbb{Z}^\bullet_{\mathcal{W}}[n] \cong F_!\mathcal{D}\left(\mathbb{Z}^\bullet_{\mathcal{W}}[n]\right) \cong F_!\mathbb{Z}^\bullet_{\mathcal{W}}[n] \cong F_*\mathbb{Z}^\bullet_{\mathcal{W}}[n],$$
where the last isomorphism follows from the fact that F is finite, and hence proper.

4. Using the self-duality of $\mathbf{I}^\bullet_X$, we find
$$j_x^!\mathbf{I}^\bullet_X \cong \mathcal{D}j_x^*\mathcal{D}\mathbf{I}^\bullet_X \cong \mathcal{D}j_x^*\mathbf{I}^\bullet_X.$$
Therefore,
$$H^k(j_x^!\mathbf{I}^\bullet_X) \cong H^k(\mathcal{D}j_x^*\mathbf{I}^\bullet_X) \cong \mathrm{Hom}(H^{-k}(j_x^*\mathbf{I}^\bullet_X), \mathbb{Z}) \oplus \mathrm{Ext}(H^{-k+1}(j_x^*\mathbf{I}^\bullet_X), \mathbb{Z}).$$
Hence, using our earlier description of the stalk cohomology, we find
$$H^k(j_x^!\mathbf{I}^\bullet_X) \cong \begin{cases} \mathbb{Z}^{m(x)}, & \text{if } k = n; \\ 0, & \text{otherwise.} \end{cases}$$

5. There is always a canonical morphism of perverse sheaves from the shifted constant sheaf to intersection cohomology with the (shifted) constant local system, i.e., a canonical morphism $\mathbb{Z}^\bullet_X[n] \xrightarrow{c} \mathbf{I}^\bullet_X$.
Because we are using $\mathbb{Z}$ as our base ring, instead of a field, $\mathbf{I}^\bullet_X$ is **not** a simple object in the Abelian category of perverse sheaves of $\mathbb{Z}$-modules. However, $\mathbf{I}^\bullet_X$ is nonetheless the intermediate extension of the constant sheaf on X_{reg}, and so has no non-trivial sub-perverse sheaves or quotient perverse sheaves with support contained in ΣX. Therefore, since our morphism induces an isomorphism when restricted to X_{reg}, its cokernel must be zero, i.e., the morphism c is a surjection.
The description of the induced map on the stalks follows at once from the fact that
$$\mathbf{I}^\bullet_X = F_*\mathbb{Z}^\bullet_{\mathcal{W}}[n] \cong F_*F^*\mathbb{Z}^\bullet_X[n].$$
$\square$

As an immediate corollary to Item (1) above, we have the well-known:

Corollary 3.3. *There is a containment $D \subseteq \Sigma X$.*

The containment above can easily be strict; this is, for instance, the case when one parameterizes the cusp.

Remark 3.4. We wish to make the costalk cohomology of a complex of sheaves more intuitive for the reader. We continue with the notation j_x from the proposition.

Let $B_\epsilon^\circ(x)$ denote an open ball of radius ϵ, centered at $x \in X$. Suppose that $\mathbf{A}^\bullet$ is a bounded constructible complex of sheaves on X, and recall that $K_{X,x}$ denotes the real link of X at x.

Then, the cohomology of $j_x^!\mathbf{A}^\bullet$ is isomorphic to the hypercohomology of a pair:

$$H^k\left(j_x^!\mathbf{A}^\bullet\right) \cong \mathbb{H}^k\left(B_\epsilon^\circ(x)\cap X, \left(B_\epsilon^\circ(x)-\{x\}\right)\cap X;\ \mathbf{A}^\bullet\right),$$

for $0 < \epsilon \ll 1$, and there exists the usual long exact sequence for this pair, in which

$$\mathbb{H}^k\left(\left(B_\epsilon^\circ(x)-\{x\}\right)\cap X;\ \mathbf{A}^\bullet\right) \cong \mathbb{H}^k\left(K_{X,x};\ \mathbf{A}^\bullet\right).$$

In particular,

$$H^k\left(j_x^!\mathbb{Z}_X^\bullet[n]\right) \cong \widetilde{H}^{n+k-1}(K_{X,x};\mathbb{Z}).$$

Note that, as $\mathbb{Z}_X^\bullet[n]$ is a perverse sheaf, $H^k\left(j_x^!\mathbb{Z}_X^\bullet[n]\right) = 0$ for $k \leq -1$. This is the cohomological manifestation of the fact that the real link of X is $(n-2)$-connected (see [Mi]).

§4. The Multiple-Point Complex

We let $\mathbf{N}^\bullet$ denote the kernel of the morphism c from Property 5 in Proposition 3.2, so that, in the Abelian category of perverse sheaves, we have a short exact sequence (which corresponds to a distinguished triangle in the derived category):

$$(\dagger) \qquad 0 \to \mathbf{N}^\bullet \to \mathbb{Z}_X^\bullet[n] \xrightarrow{c} \mathbf{I}_X^\bullet \to 0.$$

In our current setting, the morphism c is particularly simple to describe on the level of stalk cohomology. Since

$$\mathbf{I}_X^\bullet = F_*\mathbb{Z}_{\mathcal{W}}^\bullet[n] \cong F_*F^*\mathbb{Z}_X^\bullet[n],$$

the morphism c agrees with the natural map

$$\mathbb{Z}_X^\bullet[n] \xrightarrow{c} F_*F^*\mathbb{Z}_X^\bullet[n].$$

On the stalk cohomology at x, this is just the diagonal inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}^{m(x)}$ in the only non-zero degree, $-n$. From the long exact sequence on stalk cohomology for our short exact sequence, we conclude that the perverse sheaf $\mathbf{N}^\bullet$ has stalk cohomology given by

$$H^k(\mathbf{N}^\bullet)_x \cong \begin{cases} \mathbb{Z}^{m(x)-1}, & \text{if } k = -n+1; \\ 0, & \text{otherwise.} \end{cases}$$

In particular, the support of $\mathbf{N}^\bullet$ is D. Note that the stalk cohomology of $\mathbf{N}^\bullet$ at $\mathbf{0}$ is $\mathbb{Z}^{r-1}$.

Remark 4.1. The reader may be wondering why the morphism c has a non-zero kernel in the category of perverse sheaves. After all, on the level of stalks, the map c induces inclusions; it may seem as though c should have a non-trivial cokernel, not kernel.

It is true that there is a complex of sheaves $\mathbf{C}^\bullet$ and a distinguished triangle in the derived category

$$\mathbb{Z}^\bullet_X[n] \xrightarrow{c} \mathbf{I}^\bullet_X \to \mathbf{C}^\bullet \xrightarrow{[1]} \mathbb{Z}^\bullet_X[n]$$

in which the stalk cohomology of $\mathbf{C}^\bullet$ is non-zero only in degree $-n$ and, in degree $-n$, is isomorphic to the cokernel of map induced on the stalks by c. However, the complex $\mathbf{C}^\bullet$ is **not** perverse; it is supported on a set of dimension $n-1$ and has non-zero cohomology in degree $-n$.

However, we can "turn" the triangle to obtain a distinguished triangle

$$\mathbf{C}^\bullet[-1] \to \mathbb{Z}^\bullet_X[n] \xrightarrow{c} \mathbf{I}^\bullet_X \xrightarrow{[1]} \mathbf{C}^\bullet,$$

where $\mathbf{C}^\bullet[-1]$ is, in fact, perverse. Thus, in the Abelian category of perverse sheaves $\mathbf{N}^\bullet := \mathbf{C}^\bullet[-1]$ is the kernel of the morphism c.

Definition 4.2. We refer to the perverse sheaf $\mathbf{N}^\bullet$ as the *multiple-point complex* (of F on X).

We want to list the important features of the multiple-point complex which we have already discussed.

Theorem 4.3. *The multiple-point complex* $\mathbf{N}^\bullet$ *has the following properties:*

1. *There is a short exact sequence in the Abelian category of perverse sheaves on X:*

$$(\ddagger) \qquad 0 \to \mathbf{N}^\bullet \to \mathbb{Z}_X^\bullet[n] \xrightarrow{c} F_*\mathbb{Z}_{\mathcal{W}}^\bullet[n] \to 0.$$

   In particular, $\mathbf{N}^\bullet$ is a perverse sheaf.

2. *The support of $\mathbf{N}^\bullet$ is D. This implies that $\mathbf{N}^\bullet$ is the extension by zero of the perverse sheaf $\mathbf{N}^\bullet_{|_D}$ to all of X.*

3. *The stalk cohomology of $\mathbf{N}^\bullet$ at a point $x \in D$ is zero in all degrees other than $-n+1$, and*

$$H^{-n+1}(\mathbf{N}^\bullet)_x \cong H^0(\mathbf{N}^\bullet[-n+1])_x \cong \mathbb{Z}^{m(x)-1}.$$

   In particular, in degree 0, the stalk cohomology of $\mathbf{N}^\bullet[-n+1]$ at the origin is $\mathbb{Z}^{r-1}$.

4. *The costalk cohomology of $\mathbf{N}^\bullet$ is given by, for all $x \in X$,*

$$H^k(j_x^!\mathbf{N}^\bullet) \cong \begin{cases} \widetilde{H}^{n+k-1}(K_{X,x};\mathbb{Z}), & \text{if } 0 \le k \le n-1; \\ 0, & \text{otherwise.} \end{cases}$$

Corollary 4.4. *D is purely $(n-1)$-dimensional (which includes the possibility of being empty).*

Proof. This is immediate from D being the support of a perverse sheaf which, on an open dense set of D, has non-zero stalk cohomology precisely in degree $-n+1$. □

Example 4.5. In general, the data about the stalk and costalk cohomology does not determine $\mathbf{N}^\bullet$; the cohomology sheaves typically would yield non-trivial local systems on the strata.

However, we can give a simple example using a hyperplane arrangement where it is easy to describe $\mathbf{N}^\bullet$. The arrangement that we use will be the most **non-generic** one possible.

Let X be a union of r distinct hyperplanes through the origin in $\mathbb{C}^5$, where all of the hyperplanes contain $\mathbb{C}^3 \times \{(0,0)\}$. Using (x,y,z,v,w) for coordinates on $\mathbb{C}^5$, this means that $X = V(g)$, where

$$g(x,y,z,v,w) = \prod_{i=1}^{r}(a_i v + b_i w)$$

and $(a_i : b_i)$ are distinct points in $\mathbb{P}^1$. This describes a product arrangement obtained by taking an arrangement of lines through the origin in $\mathbb{C}^2$ and taking the product with $\mathbb{C}^3$.

Such an X is parameterized by using r disjoint copies of $\mathbb{C}^4$ for the domain of F. We use $({}^iu_0, {}^iu_1, {}^iu_2, {}^iu_3)$ for coordinates on the ith copy of $\mathbb{C}^4$, and let F_i be the restriction of F to this copy. Then,

$$F_i({}^iu_0, {}^iu_1, {}^iu_2, {}^iu_3) := ({}^iu_0, {}^iu_1, {}^iu_2,\, b_i{}^iu_3, -a_i{}^iu_3).$$

Note that D, the support of $\mathbf{N}^\bullet$, is $\mathbb{C}^3 \times \{(0,0)\}$. Furthermore, the product structure on X makes the description of $\mathbf{N}^\bullet$ easy; $\mathbf{N}^\bullet$ is the shifted extension by zero of of the constant sheaf on D with stalks $\mathbb{Z}^{r-1}$; to be precise, if we let j denote the inclusion of D into X, then

$$\mathbf{N}^\bullet \cong j_!\left(\mathbb{Z}^\bullet_D\right)^{r-1}[3].$$

We will now define a special notion of reduced hypercohomology, which we will use throughout the remainder of this paper. The properties described are given by a technical lemma, which we prove in the appendix, Section 7.

Note that, although $\mathbf{N}^\bullet$ is perverse, we will usually use the shifted complex $\mathbf{N}^\bullet[-n+1]$ so that the non-zero stalk cohomology is in degree 0; the purpose of this is to make $\mathbf{N}^\bullet[-n+1]$ easier to think of as being analogous to constant $\mathbb{Z}$ coefficients, but with multiplicities.

We remind the reader of our earlier convention: the reduced cohomology of the empty set, $\widetilde{H}^k(\emptyset;\mathbb{Z})$, is zero in all degrees other than degree -1, and $\widetilde{H}^{-1}(\emptyset;\mathbb{Z}) = \mathbb{Z}$.

Definition 4.6. We define the $(r-1)$-*reduced hypercohomology*

$$\widetilde{\mathbb{H}}^k(M_{h,\mathbf{0}} \cap D; \mathbf{N}^\bullet[-n+1])$$

to be $H^k(\phi_h\mathbf{N}^\bullet[-n+1])_{\mathbf{0}}$ and note that this is justified by Lemma 7.1, since, with this definition,

- If $k \neq -1$ or 0, then

$$\widetilde{\mathbb{H}}^k(M_{h,\mathbf{0}} \cap D; \mathbf{N}^\bullet[-n+1]) \cong \mathbb{H}^k(M_{h,\mathbf{0}} \cap D; \mathbf{N}^\bullet[-n+1]).$$

- There is an equality of Euler characteristics

$$\chi\left(\widetilde{\mathbb{H}}^*(M_{h,\mathbf{0}} \cap D; \mathbf{N}^\bullet[-n+1])\right) = \chi\left(\mathbb{H}^k(M_{h,\mathbf{0}} \cap D; \mathbf{N}^\bullet[-n+1])\right) - (r-1).$$

- If $\dim_{\mathbf{0}} D \cap V(h) \le n-2$, then $\widetilde{\mathbb{H}}^{-1}(M_{h,\mathbf{0}} \cap D; \mathbf{N}^\bullet[-n+1]) = 0$ and

  $\operatorname{rank} \widetilde{\mathbb{H}}^0(M_{h,\mathbf{0}} \cap D; \mathbf{N}^\bullet[-n+1]) = \operatorname{rank} \mathbb{H}^0(M_{h,\mathbf{0}} \cap D; \mathbf{N}^\bullet[-n+1]) - (r-1).$

- If $r = 1$, then $\widetilde{\mathbb{H}}^{-1}(M_{h,\mathbf{0}} \cap D; \mathbf{N}^\bullet[-n+1]) = 0$ and

$$\widetilde{\mathbb{H}}^0(M_{h,\mathbf{0}} \cap D; \mathbf{N}^\bullet[-n+1]) \cong \mathbb{H}^0(M_{h,\mathbf{0}} \cap D; \mathbf{N}^\bullet[-n+1]).$$

The following theorem is now easy to prove.

Theorem 4.7. *There is a long exact sequence, relating the Milnor fiber of h, the Milnor fibers of $h \circ F$, and the hypercohomology of the Milnor fiber of h restricted to D with coefficients in $\mathbf{N}^\bullet$, given by*

$$\cdots \to \widetilde{\mathbb{H}}^{j-1}(D \cap M_{h,\mathbf{0}}; \mathbf{N}^\bullet[-n+1]) \to \widetilde{H}^j(M_{h,\mathbf{0}}; \mathbb{Z}) \to$$
$$\bigoplus_i \widetilde{H}^j(M_{h\circ F, p_i}; \mathbb{Z}) \to \widetilde{\mathbb{H}}^j(D \cap M_{h,\mathbf{0}}; \mathbf{N}^\bullet[-n+1]) \to \cdots \ .$$

This long exact sequence is compatible with the Milnor monodromy automorphisms in each degree.

Proof. We apply the exact functor $\phi_h[-1]$ to the short exact sequence (‡) which defines $\mathbf{N}^\bullet$ to obtain the following short exact sequence of perverse sheaves:

$$0 \to \phi_h[-1]\mathbf{N}^\bullet \to \phi_h[-1]\mathbb{Z}^\bullet_X[n] \xrightarrow{\hat{c}} \phi_h[-1]F_*\mathbb{Z}^\bullet_{\mathcal{W}}[n] \to 0,$$

where $\hat{c} = \phi_h[-1]c$. As the Milnor monodromy automorphism is natural, the maps in this short exact sequence commute with the Milnor monodromies.

If we let $\widehat{F}$ denote the restriction of F to a map from $(h \circ F)^{-1}(0)$ to $h^{-1}(0)$, then there is the well-known natural base change isomorphism (see Exercise VIII.15 of [KS] or Proposition 4.2.11 of [D]):

$$\phi_h[-1]F_*\mathbb{Z}^\bullet_{\mathcal{W}}[n] \cong \widehat{F}_*\phi_{h\circ F}[-1]\mathbb{Z}^\bullet_{\mathcal{W}}[n].$$

By the induced long exact sequence on stalk cohomology and the lemma, we are finished. □

Example 4.8. To demonstrate Theorem 4.7, we will select a function h and use the X, g, F, and $\mathbf{N}^\bullet$ given in Example 4.5; this was the example of a very non-generic hyperplane arrangement in $\mathbb{C}^5$. However, to be more concrete, we will fix $r = 3$ and choose specific (a_i, b_i). Note that $n = 4$ in this example.

We let $X = V(g)$, where

$$g(x, y, z, v, w) = vw(v + 2w);$$

hence, $D = \mathbb{C}^3 \times \{(0, 0)\}$.

The finite map F is given by its restrictions F_i, $i = 1, 2, 3$, to each of the copies of $\mathbb{C}^4$:

$$F_1({}^1u_0, {}^1u_1, {}^1u_2, {}^1u_3) := ({}^1u_0, {}^1u_1, {}^1u_2, 0, -{}^1u_3),$$

$$F_2({}^2u_0, {}^2u_1, {}^2u_2, {}^2u_3) := ({}^2u_0, {}^2u_1, {}^2u_2, {}^2u_3, 0),$$

and

$$F_3({}^3u_0, {}^3u_1, {}^3u_2, {}^3u_3) := ({}^3u_0, {}^3u_1, {}^3u_2, 2({}^3u_3), -{}^3u_3),$$

We saw earlier that $\mathbf{N}^\bullet$ is the extension by zero of $(\mathbb{Z}^\bullet_D)^2\,[3]$ to all of X, so that $\mathbf{N}^\bullet[-n+1] \cong \mathbf{N}^\bullet[-4+1]$ is the extension by zero of $(\mathbb{Z}^\bullet_D)^2$.

Now, let $h : X \to \mathbb{C}$ be given by

$$h(x, y, z, v, w) := z^2 - y^3 - xy^2 + v^2 - w^2.$$

Note that $\dim_0 D \cap V(h) = 2 = n - 2$ so that, as described in Definition 4.6, $\widetilde{\mathbb{H}}^{-1}(M_{h,\mathbf{0}} \cap D; \mathbf{N}^\bullet[-3]) = 0$.

The space $M_{h,\mathbf{0}} \cap D$ is the Milnor fiber of $\tilde{h} := h_{|_D}$, which is given by

$$\tilde{h}(x, y, z) = z^2 - y^3 - xy^2,$$

where we have identified $\mathbb{C}^3 \times \{(0, 0)\}$ with $\mathbb{C}^3$. The zero locus of the function $\tilde{h}$ is the Whitney umbrella, presented as a family of nodes degenerating to a cusp, and it is well-known that the Milnor fiber of this $\tilde{h}$ has the homotopy-type of a single 2-dimensional sphere.

We also see that

$$(h \circ F_1)({}^1u_0, {}^1u_1, {}^1u_2, {}^1u_3) = ({}^1u_2)^2 - ({}^1u_1)^3 - {}^1u_0({}^1u_1)^2 - ({}^1u_3)^2,$$

$$(h \circ F_2)({}^2u_0, {}^2u_1, {}^2u_2, {}^2u_3) = ({}^2u_2)^2 - ({}^2u_1)^3 - {}^2u_0({}^2u_1)^2 + ({}^2u_3)^2,$$

and

$$(h \circ F_3)({}^3u_0, {}^3u_1, {}^3u_2, {}^3u_3) = ({}^3u_2)^2 - ({}^3u_1)^3 - {}^3u_0({}^3u_1)^2 + 3({}^3u_3)^2.$$

All three of these are what are known as "suspensions" of the Whitney umbrella; that is, we have the function defining the Whitney umbrella and then we add a constant times the square of a new variable. By the result of Sebastiani and Thom in [ST], the Milnor fiber at the origin of each of the $h \circ F_i$ has the homotopy-type of a single 3-dimensional sphere.

Putting all of this together, we find that the only non-zero portion of the long exact sequence from Theorem 4.7 is:

$$\bigoplus_i \widetilde{H}^2(M_{h\circ F,p_i};\mathbb{Z}) = 0 \to \mathbb{Z}^2 \to \widetilde{H}^3(M_{h,\mathbf{0}};\mathbb{Z}) \to \mathbb{Z}^3 \to 0.$$

Therefore, $\widetilde{H}^3(M_{h,\mathbf{0}};\mathbb{Z}) \cong \mathbb{Z}^5$, while $\widetilde{H}^j(M_{h,\mathbf{0}};\mathbb{Z}) = 0$ for $j \neq 3$.

Theorem 4.7 has the following corollary, which gives us a refinement as to why one should think of

$$H^k(\phi_h \mathbf{N}^\bullet[-n+1])_{\mathbf{0}}$$

as the $(r-1)$-reduced hypercohomology of $\mathbb{H}^k(M_{h,\mathbf{0}} \cap D; \mathbf{N}^\bullet[-n+1])$.

Corollary 4.9. *Suppose that* $\dim_{\mathbf{0}} \Sigma_{\mathrm{top}} h \leq n-2$. *Then,*

$$\widetilde{\mathbb{H}}^{-1}(D \cap M_{h,\mathbf{0}}; \mathbf{N}^\bullet[-n+1]) = 0$$

and $\widetilde{\mathbb{H}}^0(D \cap M_{h,\mathbf{0}}; \mathbf{N}^\bullet[-n+1])$ *is free Abelian.*

Consequently, $\widetilde{\mathbb{H}}^0(D \cap M_{h,\mathbf{0}}; \mathbf{N}^\bullet[-n+1])$ *is obtained from*

$$\mathbb{H}^0(D \cap M_{h,\mathbf{0}}; \mathbf{N}^\bullet[-n+1])$$

by removing $r-1$ *direct summands of* $\mathbb{Z}$.

Proof. Note that $\dim_{\mathbf{0}} \Sigma_{\mathrm{top}} h \leq n-2$ implies that $\dim_{\mathbf{0}} V(h) < n$. Now, since $\dim_{\mathbf{0}} V(h) < n$, $\bigoplus_i \widetilde{H}^{-1}(M_{h\circ F,p_i};\mathbb{Z}) = 0$, and part of the long exact sequence from the theorem is

$$0 \to \widetilde{\mathbb{H}}^{-1}(D \cap M_{h,\mathbf{0}}; \mathbf{N}^\bullet[-n+1]) \to \widetilde{H}^0(M_{h,\mathbf{0}};\mathbb{Z}) \to$$
$$\bigoplus_i \widetilde{H}^0(M_{h\circ F,p_i};\mathbb{Z}) \to \widetilde{\mathbb{H}}^0(D \cap M_{h,\mathbf{0}}; \mathbf{N}^\bullet[-n+1]) \to H^1(M_{h,\mathbf{0}};\mathbb{Z}) \to \cdots.$$

Each $\widetilde{H}^0(M_{h\circ F,p_i};\mathbb{Z})$ is free Abelian and the Universal Coefficient Theorem for cohomology tells us that $H^1(M_{h,\mathbf{0}};\mathbb{Z})$ is free Abelian.

Since $\dim_{\mathbf{0}} \Sigma_{\mathrm{top}} h \leq n-2$, Proposition 2.1 tells us that $\widetilde{H}^0(M_{h,\mathbf{0}};\mathbb{Z}) = 0$, and we immediately conclude that

$$\widetilde{\mathbb{H}}^{-1}(D \cap M_{h,\mathbf{0}}; \mathbf{N}^\bullet[-n+1]) = 0$$

and $\widetilde{\mathbb{H}}^0(D \cap M_{h,\mathbf{0}}; \mathbf{N}^\bullet[-n+1])$ is free Abelian. The final conclusion follows now from the splitting of the exact sequence in Item 3 of Lemma 7.1. $\square$

Corollary 4.10. *If $S \cap \Sigma(h \circ F) = \emptyset$, then there is an isomorphism*

$$\widetilde{H}^j(M_{h,\mathbf{0}}; \mathbb{Z}) \cong \widetilde{\mathbb{H}}^{j-1}(D \cap M_{h,\mathbf{0}}; \mathbf{N}^\bullet[-n+1])$$

and this isomorphism commutes with the Milnor monodromies.

Example 4.11. Suppose that we have a finite map $f : (\mathcal{V}, S) \to (\Omega, \mathbf{0})$, where $\mathcal{V}$ and Ω are open neighborhoods of S in $\mathbb{C}^d$ and of the origin in $\mathbb{C}^{d+1}$, respectively. Suppose that $\mathcal{T}$ is an open neighborhood of the origin in $\mathbb{C}^d$, and that $F : \mathcal{T} \times \mathcal{V} \to \mathcal{T} \times \Omega$ is an unfolding of $f = f_{\mathbf{0}}$, i.e., F is a finite analytic map of the form $F(\mathbf{t}, \mathbf{v}) = (\mathbf{t}, f_{\mathbf{t}}(\mathbf{v}))$, where, for each $\mathbf{t} \in \mathcal{T}$, $f_{\mathbf{t}}$ is a finite map from $\mathcal{V}$ to Ω.

Let X denote the image of F, continue to write F for the map from $\mathcal{T} \times \mathcal{V}$ to X, and let h be the projection onto the first coordinate; thus, $(h \circ F)(t_1, \dots, t_d, \mathbf{v}) = t_1$. Then, $S \cap \Sigma(h \circ F) = \emptyset$ and so $\widetilde{H}^j(M_{h,\mathbf{0}}; \mathbb{Z})$ is isomorphic to

$$\widetilde{\mathbb{H}}^{j-1}(D \cap M_{h,\mathbf{0}}; \mathbf{N}^\bullet[-n+1])$$

by an isomorphism which commutes with the Milnor monodromies.

Before we can prove the next corollary, we need to recall a lemma, which is well-known to experts in the field. See, for instance, [D], Theorem 4.1.22 (note that the setting of [D], Theorem 4.1.22, is algebraic, but that assumption is used in the proof only to guarantee that there are a finite number of strata).

Lemma 4.12. *Let $\mathfrak{S}$ be a complex analytic Whitney stratification, with connected strata, of a complex analytic space Y. Suppose that $\mathfrak{S}$ contains a finite number of strata. Let $\mathbf{A}^\bullet$ be a bounded complex of $\mathbb{Z}$-modules which is constructible with respect to $\mathfrak{S}$. For each stratum S, let p_S denote a point in S.*

Then, there is the following additivity/multiplicativity formula for the Euler characteristics:

$$\chi\left(\mathbb{H}^*(Y; \mathbf{A}^\bullet)\right) = \sum_S \chi(S)\chi(\mathbf{A}^\bullet)_{p_S}.$$

Corollary 4.13. *The relationship between the reduced Euler characteristics of the Milnor fiber of h at $\mathbf{0}$, the Milnor fibers of $h \circ F$, and the X_k's is given by*

$$\widetilde{\chi}(M_{h,\mathbf{0}}) = r - 1 + \sum_i \widetilde{\chi}(M_{h \circ F, p_i}) - \sum_{k \geq 2} (k-1)\chi(X_k \cap M_{h,\mathbf{0}}).$$

Proof. Via additivity of the Euler characteristic in the hypercohomology long exact sequence given in Theorem 4.7, we obtain the following relation:

$$\begin{aligned}\widetilde{\chi}(M_{h,\mathbf{0}}) &= \sum_i \widetilde{\chi}\left(M_{h\circ F,p_i}\right) - \chi\left(\widetilde{\mathbb{H}}^*(D\cap M_{h,\mathbf{0}};\mathbf{N}^\bullet[-n+1])\right)\\ &= r-1+\sum_i \widetilde{\chi}\left(M_{h\circ F,p_i}\right) - \chi\left(\mathbb{H}^*(D\cap M_{h,\mathbf{0}};\mathbf{N}^\bullet[-n+1])\right).\end{aligned}$$

We are then finished, provided that we show that

$$\chi(D\cap M_{h,\mathbf{0}};\mathbf{N}^\bullet[-n+1]) \;=\; \sum_{k\geq 2}(k-1)\chi(X_k\cap M_{h,\mathbf{0}}).$$

For this, we use Lemma 4.12. Take a complex analytic Whitney stratification $\mathfrak{S}'$ of D such that $\mathbf{N}^\bullet_{|_D}$ is constructible with respect to $\mathfrak{S}'$; hence, for each k, $D\cap X_k$ is a union of strata. As $M_{h,\mathbf{0}}$ transversely intersects these strata, there is an induced Whitney stratification $\mathfrak{S}=\{S\}$ on $D\cap M_{h,\mathbf{0}}$ and also on each $D\cap X_k\cap M_{h,\mathbf{0}}$; these stratifications have a finite number of strata, since the Milnor fiber is defined inside a small ball and $\mathfrak{S}'$ is locally finite.

Now, since the Euler characteristic of the stalk cohomology of $\mathbf{N}^\bullet[-n+1]$ at a point $x\in X_k$ is $(k-1)$, Lemma 4.12 yields

$$\chi(D\cap M_{h,\mathbf{0}};\mathbf{N}^\bullet[-n+1]) = \sum_k \sum_{S\subseteq D\cap X_k\cap M_{h,\mathbf{0}}}(k-1)\chi(S).$$

Finally, , we "put back together" the Euler characteristics of the X_k's, i.e.,

$$\chi(X_k\cap M_{h,\mathbf{0}}) = \sum_{S\subseteq D\cap X_k\cap M_{h,\mathbf{0}}}\chi(S),$$

by again applying Lemma 4.12 to the constant sheaf on $X_k\cap M_{h,\mathbf{0}}$. □

Remark 4.14. We did not need to use $\mathbf{N}^\bullet$ to prove the above corollary. It follows quickly from the base change isomorphism which appears in the proof of Theorem 4.7, but, having the theorem, it seems natural to use it in the proof.

Intuitively, the base change isomorphism yields the same thing as the generalized branched covering argument which we described in the Introduction. However, there are some technical details that must be dealt with using this branched covering approach which are avoided by appealing to the base change isomorphism.

§5. The Isolated Critical Point Case

The case where $\mathbf{0}$ is an isolated point in $\Sigma_{\rm top}h$ is of particular interest.

Theorem 5.1. *Suppose that $\mathbf{0}$ is an isolated point in $\Sigma_{\rm top}h$. Then,*

1. *for all $p_i \in S$, $\dim_{p_i} \Sigma(h\circ F) \leq 0$,*
2. *$\widetilde{\mathbb{H}}^*(D\cap M_{h,\mathbf{0}}; \mathbf{N}^\bullet[-n+1])$ is non-zero in (at most) one degree, degree $n-2$, where it is free Abelian, and*
3. *the reduced, integral cohomology of $M_{h,\mathbf{0}}$ is non-zero in, at most, one degree, degree $n-1$, where it is free Abelian of rank*

$$\begin{aligned}\mu_{\mathbf{0}}(h) &= \Big[\sum_i \mu_{p_i}(h\circ F)\Big] + \operatorname{rank}\widetilde{\mathbb{H}}^{n-2}(D\cap M_{h,\mathbf{0}}; \mathbf{N}^\bullet[-n+1]) \\ &= \Big[\sum_i \mu_{p_i}(h\circ F)\Big] + (-1)^{n-1}\Big[(r-1) - \sum_{k\geq 2}(k-1)\chi(X_k\cap M_{h,\mathbf{0}})\Big].\end{aligned}$$

4. *In particular, if $\mathbf{0}$ is an isolated point in $\Sigma_{\rm top}h$ and $S\cap\Sigma(h\circ F) = \emptyset$, then*

$$\mu_{\mathbf{0}}(h) = \operatorname{rank}\widetilde{\mathbb{H}}^{n-2}(D\cap M_{h,\mathbf{0}}; \mathbf{N}^\bullet[-n+1]) = (-1)^{n-1}\Big[(r-1) - \sum_{k\geq 2}(k-1)\chi(X_k\cap M_{h,\mathbf{0}})\Big].$$

Proof. Except for the last equalities in each line, this follows from Proposition 2.1 and $(*)$ in the proof of Theorem 4.7, since the hypothesis is equivalent to $\mathbf{0}$ being an isolated point in the support of $\phi_h[-1]\mathbb{Z}_X[n]$, and perverse sheaves which are supported at just an isolated point have non-zero stalk cohomology in only one degree, namely degree 0.

The final equalities in each line follow from Corollary 4.13. □

Example 5.2. Let us return to the unfolding situation in Example 4.11, but now suppose that F is a stable unfolding of f with an isolated instability. Then, as before, letting h be a projection onto an unfolding coordinate, $\mathbf{0}$ is an isolated point in $\Sigma_{\rm top}h$ and $S\cap\Sigma(h\circ F) = \emptyset$.

Thus, the stable fiber has the cohomology of a finite bouquet of $(n-1)$-spheres, where the number of spheres, the Milnor number, is given by

$$\operatorname{rank}\widetilde{\mathbb{H}}^{n-2}(D\cap M_{h,\mathbf{0}}; \mathbf{N}^\bullet[-n+1]) = (-1)^{n-1}\Big[(r-1) - \sum_{k\geq 2}(k-1)\chi(X_k\cap M_{h,\mathbf{0}})\Big].$$

Note, in particular, that this implies that the right-hand side is non-negative, which is distinctly non-obvious.

Consider the simple, but illustrative, specific example where $r = 1$, $f(u) = (u^2, u^3)$, and the stable unfolding is given by $F(t,u) = (t, u^2 - t, u(u^2 - t))$. Let X be the image of F, and let $h : X \to \mathbb{C}$ be the projection onto the first coordinate, so that $(h \circ F)(t,u) = t$. Note that, using (t,x,y) as coordinates on $\mathbb{C}^3$, we have $X = V(y^2 - x^3 - tx^2)$.

Clearly $\mathbf{0} \notin \Sigma(h \circ F)$, and $\mathbf{0}$ is an isolated point in $\Sigma_{\text{top}} h$. For $k \geq 2$, the only X_k which is not empty is X_2, which equals the t-axis minus the origin. Furthermore, $X_2 \cap M_{h,\mathbf{0}}$ is a single point.

We conclude from Theorem 5.1 that $M_{h,\mathbf{0}}$, which is the complex link of X, has the cohomology of a single 1-sphere.

As a further application, we recover a classical formula for the Milnor number, as given in Theorem 10.5 of [Mi]:

Theorem 5.3. *Suppose that $n = 2$ and that F is a one-parameter unfolding of a parameterization f of a plane curve singularity with r irreducible components at the origin. Let t be the unfolding parameter and suppose that the only singularities of $M_{t_{|_X},\mathbf{0}}$ are nodes, and that there are δ of them. Recall that $X = V(g)$, and let $g_0 := g_{|_{V(t)}}$. Then, the Milnor number of g_0 is given by the formula:*

$$\mu_{\mathbf{0}}\,(g_0) = 2\delta - r + 1.$$

Proof. We recall the following formula for the Milnor number of $g_{|_{V(t)}}$ at $\mathbf{0}$ [M3]:

$$\mu_{\mathbf{0}}\left(g_{|_{V(t)}}\right) = \left(\Gamma^1_{g,t} \cdot V(t)\right)_{\mathbf{0}} + \left(\Lambda^1_{g,t} \cdot V(t)\right)_{\mathbf{0}},$$

where $\Gamma^1_{g,t}$ is the relative polar curve of g with respect to t, and $\Lambda^1_{g,t}$ is the one-dimensional Lê cycle of g with respect to t.

Using that the only singularities of $M_{t_{|_X},\mathbf{0}}$ are nodes, we immediately have $\left(\Lambda^1_{g,t} \cdot V(t)\right)_{\mathbf{0}} = \delta$. Since the unfolding function F has an isolated instability at $\mathbf{0}$, $\mu_{\mathbf{0}}(t_{|_X})$ is equal to $\left(\Gamma^1_{g,t} \cdot V(t)\right)_{\mathbf{0}}$ (see, for example, [M1]).

Now, Corollary 4.13 tells us that

$$\mu_{\mathbf{0}}(t_{|_X}) = -r + 1 + \sum_{k \geq 2} (k-1)\chi(X_k \cap M_{t_{|_X},\mathbf{0}}).$$

By assumption, $\chi(X_2 \cap M_{t_{|_X},\mathbf{0}})$ is the only non-zero summand in the above equation, and it is immediately seen to be the number of double points of $X \cap V(t)$ appearing in a stable perturbation. Thus,

$$\mu_{\mathbf{0}}\left(g_{|_{V(t)}}\right) = 2\delta - r + 1$$

as desired. □

§6. Questions and Future Directions

Question/Comment 1: If $\mathbf{0}$ is an isolated point in $\Sigma_{\mathrm{top}} h$, then Theorem 5.1 provides a nice way of calculating the only non-zero cohomology group of the Milnor fiber of h.

However, even if $S \cap \Sigma(h \circ F) = \emptyset$, it is unclear how much effectively calculable data about the cohomology of $M_{h,\mathbf{0}}$ one can extract from Corollary 4.10 if $\dim_{\mathbf{0}} \Sigma_{\mathrm{top}} h > 0$ and $n \geq 3$ (so $\dim_{\mathbf{0}} D \geq 2$). Yes, we would know that

$$\widetilde{H}^j(M_{h,\mathbf{0}}; \mathbb{Z}) \cong \widetilde{\mathbb{H}}^{j-1}(D \cap M_{h,\mathbf{0}}; \mathbf{N}^\bullet[-n+1]),$$

but this hypercohomology on the right is highly non-trivial to calculate. There is a spectral sequence that one could hope to use, but that does not seem to yield manageable data.

So the question is: if $\dim_{\mathbf{0}} \Sigma_{\mathrm{top}} h > 0$ and $n \geq 3$, how do we say anything useful about $\widetilde{\mathbb{H}}^{j-1}(D \cap M_{h,\mathbf{0}}; \mathbf{N}^\bullet[-n+1])$?

Question/Comment 2: Even when $\mathbf{0}$ is an isolated point in $\Sigma_{\mathrm{top}} h$, it is not clear how to generalize Theorem 5.3 to the case where $n > 2$, i.e., the case in which g_0 has a singular set of dimension greater than zero.

Using the results of [M2], we can derive a formula which generalizes

$$\mu_{\mathbf{0}}\left(g_{|_{V(t)}}\right) = \left(\Gamma^1_{g,t} \cdot V(t)\right)_{\mathbf{0}} + \left(\Lambda^1_{g,t} \cdot V(t)\right)_{\mathbf{0}}$$

to produce formulas for the cohomology of $M_{g_0,\mathbf{0}}$ in all degrees, but these formulas once again refer to the hypercohomology of a certain perverse sheaf, and we have no effective means for calculating this hypercohomology.

Of course, part of the question here is: what is the "right" generalization of the condition that "the only singularities of $M_{t_{|_X},\mathbf{0}}$ are nodes"?

We cannot require that the singularities of $M_{t_{|_X},\mathbf{0}}$ be stable because the condition that $\mathbf{0}$ is an isolated point in $\Sigma_{\mathrm{top}} h$ implies that the singular set changes only on a discrete set.

Question/Comment 3: Another interesting direction of research might be to eliminate the finite map F altogether. In the setting of this paper, the fact that the stalk cohomology of $\mathbf{I}^\bullet_X$ is given by, for all $x \in X$,

$$H^k(\mathbf{I}^\bullet_X)_x \cong \begin{cases} \mathbb{Z}^{m(x)}, & \text{if } k = -n \\ 0, & \text{otherwise} \end{cases}$$

makes it seem as though it might be worthwhile to define, in general, *virtually parameterizable hypersurfaces* (VPHs) as those hypersurfaces for which the intersection cohomology has such a form. One could then study deformations of a given VPH via a family of VPHs.

§7. The $(r-1)$-reduction Lemma

In this section, we prove the lemma which we used to justify the terminology "$(r-1)$-reduced cohomology" in Definition 4.6.

Lemma 7.1.

1. *For all k,*

$$H^k(\phi_h \mathbb{Z}^\bullet_X)_{\mathbf{0}} \cong \widetilde{H}^k(M_{h,\mathbf{0}};\mathbb{Z}),$$

which is possibly non-zero only for $n-s-1 \leq k \leq n-1$, where $s := \dim_{\mathbf{0}} \Sigma_{\text{top}} h \leq n$. Furthermore, when $k=-1$, this cohomology is non-zero if and only if h is identically zero (so that $M_{h,\mathbf{0}} = \emptyset$).

2. *For all k,*

$$H^k(\phi_h F_*\mathbb{Z}^\bullet_{\mathcal{W}})_{\mathbf{0}} \cong \bigoplus_i \widetilde{H}^k(M_{h\circ F, p_i};\mathbb{Z}),$$

which is possibly non-zero only for $n-\hat{s}-1 \leq k \leq n-1$, where

$$\hat{s} := \max_i\{\dim_{p_i} \Sigma(h\circ F)\} \leq n.$$

Furthermore, when $k=-1$, this cohomology is non-zero if and only if h is identically zero on at least one irreducible component of X.

3. *$H^k(\phi_h \mathbf{N}^\bullet[-n+1])_{\mathbf{0}}$ is possibly non-zero only for $-1 \leq k \leq n-2$. Furthermore, if h is not identically zero on any irreducible component of D, i.e., if $\dim_{\mathbf{0}} D \cap V(h) \leq n-2$, then $H^{-1}(\phi_h \mathbf{N}^\bullet[-n+1])_{\mathbf{0}} = 0$.*

   We also have:

   - *For $k \neq -1$ or 0,*

   $$H^k(\phi_h \mathbf{N}^\bullet[-n+1])_{\mathbf{0}} \cong \mathbb{H}^k(M_{h,\mathbf{0}} \cap D; \mathbf{N}^\bullet[-n+1]).$$

   - *If $r=1$, then, for all k,*

   $$H^k(\phi_h \mathbf{N}^\bullet[-n+1])_{\mathbf{0}} \cong \mathbb{H}^k(M_{h,\mathbf{0}} \cap D; \mathbf{N}^\bullet[-n+1]).$$

- *There is an exact sequence*
  $0 \to H^{-1}(\phi_h[-1]\mathbf{N}^\bullet[-n+1])_{\mathbf{0}} \to \mathbb{Z}^{r-1} \to$
  $\mathbb{H}^0(M_{h,\mathbf{0}} \cap D; \mathbf{N}^\bullet[-n+1]) \to H^0(\phi_h[-1]\mathbf{N}^\bullet[-n+1])_{\mathbf{0}} \to 0.$

Proof. Let B denote a small open ball around the origin in $\mathbb{C}^{n+1}$. Then, for every bounded, constructible complex $\mathbf{A}^\bullet$ on X, if we let $Y = \operatorname{supp} \mathbf{A}^\bullet$, then

$$H^k(\phi_h[-1]\mathbf{A}^\bullet)_{\mathbf{0}} \cong \mathbb{H}^k(B\cap X, M_{h,\mathbf{0}}; \mathbf{A}^\bullet) \cong \mathbb{H}^k(B\cap Y, M_{h,\mathbf{0}}\cap Y; \mathbf{A}^\bullet),$$

where this hypercohomology group fits into the hypercohomology long exact sequence of the pair $(B\cap X, M_{h,\mathbf{0}})$.

Consequently, using $\mathbf{A}^\bullet = \mathbb{Z}^\bullet_X[n]$, we find that $H^k(\phi_h[-1]\mathbb{Z}^\bullet_X[n])_{\mathbf{0}}$ is, in fact, equal to the standard reduced cohomology of the Milnor fiber $\widetilde{H}^{k+n-1}(M_{h,\mathbf{0}};\mathbb{Z})$, **provided** that we use our convention on the reduced cohomology of the empty set.

Suppose instead that we use $\mathbf{A}^\bullet = F_*\mathbb{Z}^\bullet_{\mathcal{W}}[n]$. Then, by the base change formula [KS], $\phi_h[-1]F_*\mathbb{Z}^\bullet_{\mathcal{W}}[n]$ is naturally isomorphic to $\widehat{F}_*\phi_{h\circ F}[-1]\mathbb{Z}^\bullet_{\mathcal{W}}[n]$, where $\widehat{F}$ denotes the map induced by F from $F^{-1}h^{-1}(0)$ to $h^{-1}(0)$.

Therefore,

$$H^k(\phi_h[-1]F_*\mathbb{Z}^\bullet_{\mathcal{W}}[n])_{\mathbf{0}} \cong \bigoplus_i H^k(\phi_{h\circ F}[-1]\mathbb{Z}^\bullet_{\mathcal{W}}[n])_{p_i},$$

which, from our work above, implies that

$$H^k(\phi_h[-1]F_*\mathbb{Z}^\bullet_{\mathcal{W}}[n])_{\mathbf{0}} \cong \bigoplus_i \widetilde{H}^{k+n-1}(M_{h\circ F,p_i};\mathbb{Z}).$$

Now we need to look at the more complicated case where $\mathbf{A}^\bullet = \mathbf{N}^\bullet$. We find

$$H^k(\phi_h[-1]\mathbf{N}^\bullet)_{\mathbf{0}} \cong \mathbb{H}^k(B\cap D, M_{h,\mathbf{0}}\cap D; \mathbf{N}^\bullet),$$

and the long exact sequence of the pair, together with the fact that we know $\mathbb{H}^*(B\cap D;\mathbf{N}^\bullet) \cong H^*(\mathbf{N}^\bullet)_{\mathbf{0}}$, gives us the exact sequence

$$\cdots \to \mathbb{H}^{-n}(M_{h,\mathbf{0}}\cap D;\mathbf{N}^\bullet) \to H^{-n+1}(\phi_h[-1]\mathbf{N}^\bullet)_{\mathbf{0}} \to \mathbb{Z}^{r-1} \to \mathbb{H}^{-n+1}(M_{h,\mathbf{0}}\cap D;\mathbf{N}^\bullet) \to$$
$$H^{-n+2}(\phi_h[-1]\mathbf{N}^\bullet)_{\mathbf{0}} \to 0 \to \mathbb{H}^{-n+2}(M_{h,\mathbf{0}}\cap D;\mathbf{N}^\bullet) \to H^{-n+3}(\phi_h[-1]\mathbf{N}^\bullet)_{\mathbf{0}} \to 0 \to \cdots.$$

We claim that $\mathbb{H}^k(M_{h,\mathbf{0}}\cap D;\mathbf{N}^\bullet) = 0$ for all $k \leq -n$. This follows immediately from the fact that $\mathbb{H}^{-n}(M_{h,\mathbf{0}}\cap D;\mathbf{N}^\bullet) \cong H^{-n+1}(\psi_h[-1]\mathbf{N}^\bullet)_{\mathbf{0}}$, and $\psi_h[-1]\mathbf{N}^\bullet$ is a perverse sheaf supported on $\overline{D - V(h)} \cap V(h)$, which has dimension less than or equal to $n-2$. Since we also know that $H^k(\mathbf{N}^\bullet)_{\mathbf{0}} = 0$ for all $k \leq -n$, we conclude the following:

- For all $k \leq -n$, $H^k(\phi_h[-1]\mathbf{N}^\bullet)_\mathbf{0} \cong \mathbb{H}^k(M_{h,\mathbf{0}} \cap D; \mathbf{N}^\bullet) = 0$.
- For all $k \geq -n+3$,

$$H^k(\phi_h[-1]\mathbf{N}^\bullet)_\mathbf{0} \cong \mathbb{H}^{k-1}(M_{h,\mathbf{0}} \cap D; \mathbf{N}^\bullet) \cong \mathbb{H}^{k+n-2}(M_{h,\mathbf{0}} \cap D; \mathbf{N}^\bullet[-n+1]).$$

- We have an exact sequence

$$0 \to H^{-n+1}(\phi_h[-1]\mathbf{N}^\bullet)_\mathbf{0} \to \mathbb{Z}^{r-1} \to \mathbb{H}^{-n+1}(M_{h,\mathbf{0}} \cap D; \mathbf{N}^\bullet) \to H^{-n+2}(\phi_h[-1]\mathbf{N}^\bullet)_\mathbf{0} \to 0.$$

If $\dim_\mathbf{0} D \cap V(h) \leq n-2$, then $\phi_h[-1]\mathbf{N}^\bullet$ is a perverse sheaf which is supported on a set of dimension at most $n-2$; the stalk cohomology in degrees less than $-(n-2) = -n+2$ is zero. Therefore, in this case,

$$H^{-n+1}(\phi_h[-1]\mathbf{N}^\bullet)_\mathbf{0} \cong H^{-1}(\phi_h \mathbf{N}^\bullet[-n+1])_\mathbf{0} = 0.$$

□

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