The Lê Numbers of One-Parameter Families of Parameterized Hypersurfaces

Generalizing Milnor's Double-Point Formula

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Milnor's Result

- Suppose $(V(g_0), \mathbf{0}) \subseteq (\mathbb{C}^2, \mathbf{0})$ is plane curve singularity in $\mathbb{C}^2$, with r irreducible components at the origin.
  Then, by a well-known result of Milnor (1968), the Milnor number $\mu_{\mathbf{0}}(g_0)$ is related to the number of double points δ which occur in a generic (stable) deformation g_{t_0} of g_0 by

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- We wish to generalize this formula to deformations of hypersurfaces with codimension-one singularities.

Parameterizing Milnor's Result

More precisely, Milnor's result can be stated as follows.

Theorem (Milnor '68)

Suppose that F is a one-parameter unfolding of a parameterization f of a plane curve singularity $V(g_0)$ with r irreducible components at the origin, and set $\operatorname{im} F = V(g)$. Let t be the unfolding parameter and suppose that the only singularities of the complex link $\mathbb{L}_{V(g),\mathbf{0}}$ are nodes, and that there are δ of them. Let $g_0 := g_{|_{V(t)}}$. Then, the Milnor number of g_0 is given by the formula:

$$\mu_{\mathbf{0}}(g_0) = 2\delta - r + 1.$$

Quick Proof of Milnor's Result

- The deformation of the function g_0 is an analytic function g defining the surface $\operatorname{im} F = V(g)$ in $\mathbb{C}^3$, together with a choice of "nice" linear form t on $\mathbb{C}^3$ provided by the unfolding parameter of the parameterization. As in the theorem, we have $g_0 := g_{|_{V(t)}}$.

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- Recall the following formula for the Milnor number of g_0 at $\mathbf{0}$:

$$\mu_{\mathbf{0}}(g_0) = \left(\Gamma^1_{g,t} \cdot V(t)\right)_{\mathbf{0}} + \left(\Lambda^1_{g,t} \cdot V(t)\right)_{\mathbf{0}},$$

where $\Gamma^1_{g,t}$ **is the relative polar curve of** g **with respect to** t, and $\Lambda^1_{g,t}$ **is the one-dimensional Lê cycle of** g **with respect to** t. Note then that, for t_0 small and non-zero, we can identify the deformed curve $V(g_{t_0})$ with the complex link $\mathbb{L}_{V(g),\mathbf{0}}$ of $V(g)$ at the origin.

Quick Proof of Milnor's Result (continued)

- By assumption, the only singularities of $\mathbb{L}_{V(g),\mathbf{0}}$ are nodes, **each of which necessarily has Milnor number equal to 1**, so we immediately have $\left(\Lambda^1_{g,t} \cdot V(t)\right)_{\mathbf{0}} = \delta$. Since the unfolding function F has an isolated instability at $\mathbf{0}$, $\mu_{\mathbf{0}}(t_{|_X})$ is defined and equal **the number of 1-spheres in the homotopy-type of the complex link**, given by $\left(\Gamma^1_{g,t} \cdot V(t)\right)_{\mathbf{0}}$.

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- Now, by [H.,Massey '17],

$$\mu_{\mathbf{0}}(t_{|X}) = -r + 1 + \sum_{k \geq 2} (k-1)\chi(V(g)_k \cap \mathbb{L}_{V(g),\mathbf{0}}),$$

where $V(g)_k := \{p \in V(g) \mid |F^{-1}(p)| = k\}$.

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  where $V(g)_k := \{p \in V(g) \,|\, |F^{-1}(p)| = k\}$.
- By assumption, $\chi(V(g)_2 \cap \mathbb{L}_{V(g),\mathbf{0}})$ is the only non-zero summand in the above equation, and it is immediately seen to be the number of double points of $V(g_0)$ appearing in a stable perturbation. Thus,

$$\mu_{\mathbf{0}}\left(g_{|_{V(t)}}\right) = 2\delta - r + 1.$$

Rewriting the Formula

- **What if we didn't have such a "stable deformation" of $V(g_0)$?** That is, what if we didn't know that the origin $\mathbf{0} \in V(g_0)$ splits into δ nodes? We can still use the techniques of [H.,Massey '17] in this situation. In this case, we have

$$\mu_{\mathbf{0}}(g_0) = -r + 1 + \sum_{p \in B_\epsilon \cap V(t - t_0)} \left(\mu_p(g_{t_0}) + m(p)\right)$$

where $m(p) := |F^{-1}(p)| - 1$.

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  where $m(p) := |F^{-1}(p)| - 1$.
- **The main idea** behind generalizing Milnor's formula to higher dimensions is that, when the hypersurface $V(g)$ is parameterized, there is a natural perverse sheaf on $V(g)$ that generalizes the function $m(p)$.

Parameterizations

- Suppose that we have a surjective finite map $F : (\mathcal{W} \times \mathbb{C}, S \times \{0\}) \to (V(g), \mathbf{0})$ which is **generically one-to-one**, and is further of the form

$$F(\mathbf{z}, t) = (f_t(\mathbf{z}), t),$$

where $f_0(\mathbf{z})$ is a generically one-to-one parameterization of $V(g_0)$. Here $\mathcal{W}$ is an open subset of $\mathbb{C}^{n-1}$, and $F^{-1}(\mathbf{0}) = S$. Let $r := |S|$.

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- Then, in the Abelian category of perverse sheaves on $V(g)$, there is a canonical surjective morphism $\mathbb{Z}^\bullet_{V(g)}[n] \xrightarrow{c} F_*\mathbb{Z}^\bullet_{\mathcal{W}}[n]$. We let $\mathbf{N}^\bullet_F$ be the kernel of this morphism, so that we have a **short exact sequence of perverse sheaves**

$$0 \to \mathbf{N}^\bullet_F \to \mathbb{Z}^\bullet_{V(g)}[n] \xrightarrow{c} F_*\mathbb{Z}^\bullet_{\mathcal{W}}[n] \to 0.$$

The Multiple Point Complex

The complex $\mathbf{N}_F^\bullet$ is called the **multiple-point complex** of the parameterization F, and is supported on the image multiple-point set $D := \overline{\{x \in V(g) \mid |F^{-1}(x)| > 1\}}$.

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- It is a perverse sheaf on $V(g)$.
- It has nonzero stalk cohomology only in degree $-(n-1)$, where $n = \dim_{\mathbf{0}} V(g)$.
- In degree $-(n-1)$, the stalk cohomology is very easy to describe: for $p \in V(g)$,

  $$H^{-(n-1)}(\mathbf{N}_F^\bullet))_p \cong \mathbb{Z}^{m(p)}.$$

  where $m(p) := |F^{-1}(p)| - 1$, as before.

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- One natural generalization of the Milnor number to higher-dimensional singularities are the Lê numbers–so we will express the **Lê numbers** of the $t = 0$ slice in terms of the Lê numbers of the $t \neq 0$ slice, together with the **characteristic polar multiplicities of** $\mathbf{N}^{\bullet}_F$ (discussed below).

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- One natural generalization of the Milnor number to higher-dimensional singularities are the Lê numbers–so we will express the **Lê numbers** of the $t = 0$ slice in terms of the Lê numbers of the $t \neq 0$ slice, together with the **characteristic polar multiplicities of $\mathbf{N}^\bullet_F$** (discussed below).
- **What sort of deformation do we want?** We don't necessarily have a deformation into something as nice as double-points. We choose the notion of an **IPA-deformation**–these are deformations which, intuitively, are those where the only "interesting" behavior happens at the origin.

Characteristic Polar Multiplicities and Lê Numbers

For any perverse sheaf $\mathbf{P}^\bullet$ on an analytic subset of $\mathbb{C}^N$, **the characteristic polar multiplicities of $\mathbf{P}^\bullet$** with respect to a "nice" choice of linear forms $\mathbf{z} = (z_0, \cdots, z_s)$, denoted $\lambda^i_{\mathbf{P}^\bullet, \mathbf{z}}(p)$ (defined in [Massey '94]) are non-negative integer-valued functions that mimic the construction of the Lê numbers associated to non-isolated hypersurface singularities.

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Indeed, one has the equalities $\lambda^i_{f,\mathbf{z}}(p) = \lambda^i_{\phi_f[-1]\mathbb{Z}^\bullet_{\mathbb{C}^N}[N]}(p)$ for $0 \leq i \leq \dim_{\mathbf{0}} \Sigma f$, and all p in some open neighborhood of $\mathbf{0}$ in $\mathbb{C}^N$.

In Milnor's orginal formula, one deforms plane curves with isolated singularities. While we can handle the case of deforming hypersurfaces with codimension-one singularities, we state below for simplicity the case of deforming surfaces with curve singularities.

Theorem (H., '17)

Suppose that $F : (\mathbb{C}^2 \times \mathbb{C}, S \times \{0\}) \to (\mathbb{C}^4, \mathbf{0})$ *is a one-parameter unfolding with parameter* t*, with* $\operatorname{im} F = V(g)$ *for some* $g \in \mathcal{O}_{\mathbb{C}^4,\mathbf{0}}$*. Suppose further that a linear form* s *is chosen such that* (t, s) *is an IPA-tuple for* g *at* $\mathbf{0}$*. Then, the following formulas hold for the Lê numbers of* g_0 *with respect to* s *at* $\mathbf{0}$*: for* $0 < |t_0| \ll |s_0| \ll \epsilon \ll 1$,

$$\lambda^0_{g_0,s}(\mathbf{0}) = -\lambda^0_{\mathbf{N}^\bullet_{f_0},s}(\mathbf{0}) + \sum_{p \in B_\epsilon \cap V(t-t_0)} \left(\lambda^0_{g_{t_0},s}(p) + \lambda^0_{\mathbf{N}^\bullet_{f_{t_0}},s}(p) \right)$$

$$\lambda^1_{g_0,s}(\mathbf{0}) = \sum_{q \in B_\epsilon \cap V(t-t_0, s-s_0)} \lambda^0_{g_{t_0},s-s_0}(q).$$

Proof

- Since (t, s) is an IPA-tuple for g at $\mathbf{0}$, we have, for $0 < |t_0| \ll |s_0| \ll \epsilon \ll 1$,

$$\begin{aligned}\lambda^0_{g_0,s}(\mathbf{0}) &= \gamma^1_{g,t}(\mathbf{0}) + \lambda^1_{g,t}(\mathbf{0}) \\ &= \lambda^0_{\mathbf{N}^\bullet_F,t}(\mathbf{0}) + \sum_{p \in B_\epsilon \cap V(t - t_0)} \lambda^0_{g_{t_0},s}(p)\end{aligned}$$

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- We must now understand $\lambda^0_{\mathbf{N}^\bullet_F,t}(\mathbf{0}) := \operatorname{rank}_{\mathbb{Z}} H^0(\phi_t[-1]\mathbf{N}^\bullet_F)_{\mathbf{0}}$.

Proof, cont.

- (t, s) being an IPA-tuple for g at $\mathbf{0}$ implies that (t, s) is $\mathbf{N}^\bullet_F$-isolating at $\mathbf{0}$, so that

$$\lambda^0_{\mathbf{N}^\bullet_{F|_{V(t)}}[-1],s}(\mathbf{0}) = \lambda^0_{\mathbf{N}^\bullet_{f_0},s}(\mathbf{0}) = \lambda^1_{\mathbf{N}^\bullet_F,(t,s)}(\mathbf{0}) - \lambda^0_{\mathbf{N}^\bullet_F,t}(\mathbf{0}).$$

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- The claim then follows from the dynamic intersection property for proper intersections:

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$$\begin{aligned}\lambda^1_{\mathbf{N}^\bullet_F,(t,s)}(\mathbf{0}) &= \left(\Lambda^1_{\mathbf{N}^\bullet_F,(t,s)} \cdot V(t)\right)_{\mathbf{0}} \\ &= \sum_{p\in B_\epsilon\cap V(t-t_0)} \left(\Lambda^1_{\mathbf{N}^\bullet_F,(t,s)} \cdot V(t-t_0)\right)_p \\ &= \sum_{p\in B_\epsilon\cap V(t-t_0)} \lambda^0_{\mathbf{N}^\bullet_{f_{t_0}},s}(p),\end{aligned}$$

for $0 < |t_0| \lll \epsilon \ll 1$, where $\Lambda^1_{\mathbf{N}^\bullet_F,(t,s)}$ is the 1-dimensional characteristic polar cycle of $\mathbf{N}^\bullet_F$ with respect to (t, s).

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  suggests a sort of "**conservation of number**" property for these quantities; is there something deeper going on here?
- Our **intuition** behind the generalization of Milnor's result is that

$$\begin{matrix}\text{Change in Milnor numbers} \\ \text{from } g_0 \text{ to } g_{t_0}\end{matrix} = -\begin{pmatrix}\text{Change in char. polar mult.} \\ \text{of } \mathbf{N}^\bullet_F \text{ from } V(g_0) \text{ to } V(g_{t_0})\end{pmatrix}$$

Thank You!

IPA-Deformations [Massey, '16]

- Given an function $f : (\mathbb{C}^{n+1}, \mathbf{0}) \to (\mathbb{C}, 0)$ and a linear form $L : (\mathbb{C}^{n+1}, \mathbf{0}) \to (\mathbb{C}, 0)$, we say that f is a deformation of $f_{|_{V(L)}}$ with **isolated polar activity** at $\mathbf{0}$ if $\dim_{\mathbf{0}} \Gamma^1_{f,L} \cap V(L) \leq 0$.

Theorem

Let f and L be as above. Then, TFAE:

1. *f is an IPA-deformation of $f_{|_{V(L)}}$ at $\mathbf{0}$.*
2. *$(\mathbf{0}, d_{\mathbf{0}}L)$ is an isolated point in the intersection $\operatorname{im} dL \cap (f \circ \pi)^{-1}(0) \cap \overline{T^*_f \mathbb{C}^{n+1}}$.*
3. *$\mathbf{0}$ is an isolated point in the support of $\phi_L[-1]\mathbb{Z}^{\bullet}_{V(f)}[n]$.*

- We say that an order tuple of linear forms $(z_0, \cdots, z_k)$ is an **IPA-Tuple** for f at $\mathbf{0}$ if, for all $0 \leq 0i \leq k$, $f_{|_{V(z_0, \ldots, z_i - 1)}}$ is an IPA-deformation of $f_{|_{V(z_0, \ldots, z_i)}}$ at $\mathbf{0}$.

Characteristic Polar Multiplicities [Massey, '94]

- Given a perverse sheaf $\mathbf{P}^\bullet$ on a complex analytic subset X of $\mathbb{C}^N$, and choice of "nice" tuple of linear forms $\mathbf{z} = (z_0, \ldots, z_s)$ on $\mathbb{C}^N$ (where $\dim \operatorname{supp} \mathbf{P}^\bullet = s$), the **characteristic polar multiplicities of $\mathbf{P}^\bullet$ with respect to $\mathbf{z}$** at a point $p \in X$ are the non-negative integers

  $$\lambda^i_{\mathbf{P}^\bullet,\mathbf{z}}(p) = \operatorname{rank}_{\mathbb{Z}} H^0(\phi_{z_i - z_i(p)}[-1]\psi_{z_{i-1}-z_{i-1}(p)}[-1]\cdots\psi_{z_0 - z_0(p)}[-1]\mathbf{P}^\bullet)_p$$

  for $0 \leq i \leq s$.
- Such numbers exist more generally for objects of $D^b_{\mathbb{C}-c}(X)$, but they are slightly more cumbersome to define (and no longer need to be non-negative!)
- Why are these useful? For all $p \in X$, one has

  $$\chi(\mathbf{P}^\bullet)_p = \sum_{i=0}^{s} (-1)^i \lambda^i_{\mathbf{P}^\bullet,\mathbf{z}}(p).$$

Characteristic Polar Multiplicities and Lê Numbers

- Let $f : (\mathbb{C}^{n+1}, \mathbf{0}) \to (\mathbb{C}, 0)$ be an analytic function germ with $\dim_{\mathbf{0}} \Sigma f = s$. Then, $\phi_f[-1]\mathbb{Z}^\bullet_{\mathbb{C}^{n+1}}[n+1]$ is a perverse sheaf on $V(f)$, with support equal to the critical locus Σf.
- Then, if $\mathbf{z} = (z_0, \dots, z_s)$ is a "nice" tuple of linear forms on $\mathbb{C}^{n+1}$, the **Lê numbers of f with respect to z** are the non-negative integers

$$\lambda^i_{f,\mathbf{z}}(p) := \lambda^i_{\phi_f[-1]\mathbb{Z}^\bullet_{\mathbb{C}^{n+1}}[n+1],\mathbf{z}}(p).$$

If $\mathbf{z}$ is "nice", then these numbers are defined for all p in some open neighborhood of $\mathbf{0}$ in $\mathbb{C}^{n+1}$.

Example

If $\dim_{\mathbf{0}} \Sigma f = 0$, then the only non-zero Lê number of f is $\lambda^0_{f,z_0}(\mathbf{0})$, and we have

$$\begin{aligned}\lambda^0_{f,z_0}(\mathbf{0}) &= \operatorname{rank}_{\mathbb{Z}} H^0(\phi_{z_0}[-1]\phi_f[-1]\mathbb{Z}^{\bullet}_{\mathbb{C}^{n+1}}[n+1])_{\mathbf{0}} \\ &= \text{Milnor number of } f \text{ at } \mathbf{0}.\end{aligned}$$

Note that any non-zero linear form z_0 sufficed for this construction.

Example

If $\dim_{\mathbf{0}} \Sigma f = 1$, then the only non-zero Lê numbers of f with respect to $\mathbf{z} = (z_0, z_1)$ are $\lambda^0_{f,\mathbf{z}}(\mathbf{0})$ and $\lambda^1_{f,\mathbf{z}}(p)$ for $p \in \Sigma f$. At $\mathbf{0}$, we have

$$\lambda^1_{f,\mathbf{z}}(\mathbf{0}) = \sum_{C \subseteq \Sigma f \text{ irr.comp. at } \mathbf{0}} \overset{\circ}{\mu}_C \left(C \cdot V(z_0)\right)_{\mathbf{0}},$$

where $\overset{\circ}{\mu}_C$ denotes the generic transverse Milnor number of f along $C\backslash\{0\}$. Note that here we need z_0 such that $\dim_{\mathbf{0}} \Sigma\left(f_{|_{V(z_0)}}\right) = 0$.