Deformation Formulas for Parametrizable Hypersurfaces

Generalizing Milnor's Double-Point Formula

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Abstract

We investigate one-parameter deformations of functions on affine space whose defining hypersurfaces can be parameterized by a finite morphism that is generically one-to-one. Such hypersurfaces must necessarily have critical loci of codimension one. With the standard assumption of isolated polar activity at the origin, we are able to completely express the Lê numbers of the special fiber in terms of the Lê numbers of the generic fiber and the characteristic polar multiplicities of the multiple-point complex, a perverse sheaf naturally associated to any parameterized hypersurface. We discuss in particular the case of a family of parameterized surfaces with one-dimensional critical loci.

Milnor's Classical Result: The Double-Point Formula

Suppose $(V(g_0), \mathbf{0}) \subseteq (\mathbb{C}^2, \mathbf{0})$ is plane curve singularity in $\mathbb{C}^2$, with r irreducible components at the origin. Then, by a well-known result of Milnor (1968), the Milnor number $\mu_{\mathbf{0}}(g_0)$ is related to the number of double points δ which occur in a generic (stable) deformation of g_0 by

$$\mu_{\mathbf{0}}(g_0) = 2\delta - r + 1.$$

We achieve a quick proof of this result in [1] using the techniques below. We now wish to generalize this formula to deformations of hypersurfaces with codimension-one singularities.

In particular, we consider the case where $V(g)$ is *parameterized*. This is to say that there is a finite, generically one-to-one morphism $F : (\mathcal{W}, S) \to (\mathcal{U}, \mathbf{0})$ with $\operatorname{im} F = V(g)$ (this is equivalent to $V(g)$ having a smooth normalization).

What Would a Generalization Look Like?

- One of the restrictions of parameterizing $V(g)$ is that $V(g)$ **must have codimension-one singularities**; that is, $\operatorname{supp} \mathbf{N}^\bullet_F = D \subseteq \Sigma g$, and D is purely $(n-1)$-dimensional. So, any deformation formula we consider must be one with codimension-one singularities.
- One natural generalization of the Milnor number to higher-dimensional singularities are the **Lê numbers** $\lambda^i_{g,\mathbf{z}}$, so we will express the Lê numbers of the $t = 0$ slice in terms of the Lê numbers of the $t \neq 0$ slice, together with the **characteristic polar multiplicities of $\mathbf{N}^\bullet_F$** (discussed below).
- **What sort of deformation do we want?** We don't necessarily have a deformation into something as nice as double-points. We choose the notion of an **IPA-deformation**–these are deformations which, intuitively, are those where the only "interesting" behavior happens at the origin, and the only changes propagate outward from the origin along curves.

Parameterizations

- Suppose that we have a surjective finite map $F : (\mathcal{W} \times \mathbb{C}, S \times \{0\}) \to (V(g), \mathbf{0})$ which is **generically one-to-one**, and is further of the form

$$F(\mathbf{z}, t) = (f_t(\mathbf{z}), t),$$

  where $f_0(\mathbf{z})$ is a generically one-to-one parameterization of $V(g_0)$. Here $\mathcal{W}$ is an open subset of $\mathbb{C}^{n-1}$, and $F^{-1}(\mathbf{0}) = S$. Let $r := |S|$.
  This means that F **is a one-parameter unfolding of** f_0.
- Then, in the Abelian category of perverse sheaves on $V(g)$, there is a canonical surjective morphism $\mathbb{Z}^\bullet_{V(g)}[n] \xrightarrow{c} F_*\mathbb{Z}^\bullet_{\mathcal{W}}[n]$. We let $\mathbf{N}^\bullet_F$ be the kernel of this morphism, so that we have a **short exact sequence of perverse sheaves**

$$0 \to \mathbf{N}^\bullet_F \to \mathbb{Z}^\bullet_{V(g)}[n] \xrightarrow{c} F_*\mathbb{Z}^\bullet_{\mathcal{W}}[n] \to 0.$$

The Multiple-Point Complex

In the above short exact sequence, the kernel $\mathbf{N}^\bullet_F$ is a perverse sheaf, called the **multiple-point complex** of the parameterization F, and is supported on the image multiple-point set $D := \{x \in V(g) \mid |F^{-1}(x)| > 1\}$. This set is always purely codimension 1 inside $V(g)$.
The multiple-point complex has several useful properties:

- It is a perverse sheaf on $V(g)$.
- It has nonzero stalk cohomology only in degree $-(n-1)$, where $n = \dim_{\mathbf{0}} V(g)$.
- In degree $-(n-1)$, the stalk cohomology is very easy to describe: for $p \in V(g)$,

$$H^{-(n-1)}(\mathbf{N}^\bullet_F))_p \cong \mathbb{Z}^{m(p)}.$$

  where $m(p) := |F^{-1}(p)| - 1$, as before.

Characteristic Polar Multiplicities

For any perverse sheaf $\mathbf{P}^\bullet$ on an analytic subset of $\mathbb{C}^N$, **the characteristic polar multiplicities of $\mathbf{P}^\bullet$** with respect to a "nice" choice of linear forms $\mathbf{z} = (z_0, \cdots, z_s)$, denoted $\lambda^i_{\mathbf{P}^\bullet,\mathbf{z}}(p)$ (defined in [2]) are non-negative integer-valued functions that mimic the construction of the **Lê numbers** $\lambda^i_{g,\mathbf{z}}$ associated to non-isolated hypersurface singularities.
More precisely, one has at a point $p \in X$ the non-negative integers

$$\lambda^i_{\mathbf{P}^\bullet,\mathbf{z}}(p) = \operatorname{rank}_{\mathbb{Z}} H^0(\phi_{z_i - z_i(p)}[-1]\psi_{z_{i-1}-z_{i-1}(p)}[-1] \cdots \psi_{z_0 - z_0(p)}[-1]\mathbf{P}^\bullet)_p,$$

for $0 \le i \le s$, where $s = \dim_{\mathbf{0}} \operatorname{supp} \mathbf{P}^\bullet$.
Such numbers exist more generally for objects of $D^b_{\mathbb{C}-c}(X)$, but they are slightly more cumbersome to define (and no longer need to be non-negative).
Indeed, one has the equalities $\lambda^i_{f,\mathbf{z}}(p) = \lambda^i_{\phi_f[-1]\mathbb{Z}^\bullet_{\mathbb{C}^N}[N]}(p)$ for $0 \le i \le \dim_{\mathbf{0}} \Sigma f$, and all p in some open neighborhood of $\mathbf{0}$ in $\mathbb{C}^N$.

In Milnor's orginal formula, one deforms plane curves with isolated singularities. While we can handle the case of deforming hypersurfaces with codimension-one singularities, we state below for simplicity the case of deforming surfaces with curve singularities.

Theorem [H, 2017]

Suppose that $F : (\mathbb{C}^2 \times \mathbb{C}, S \times \{0\}) \to (\mathbb{C}^4, \mathbf{0})$ is a one-parameter unfolding with parameter t, with $\operatorname{im} F = V(g)$ for some $g \in \mathcal{O}_{\mathbb{C}^4,\mathbf{0}}$. Suppose further that (t, s) is an IPA-tuple for g at $\mathbf{0}$. Then, the following formulas hold for the Lê numbers of $g_{|_{V(t)}}$ with respect to s at $\mathbf{0}$:

$$\lambda^0_{g_{|_{V(t)}},s}(\mathbf{0}) = -\lambda^0_{\mathbf{N}^\bullet_{F_0},s}(\mathbf{0}) + \sum_{p \in B_\epsilon \cap V(t-t_0)} \left(\lambda^0_{g_{|_{V(t-t_0)}},s}(p) + \lambda^0_{\mathbf{N}^\bullet_{F_{t_0}},s}(p) \right)$$

$$\lambda^1_{g_{|_{V(t)}},s}(\mathbf{0}) = \sum_{q \in B_\epsilon \cap V(t-t_0, s-s_0)} \lambda^0_{g_{|_{V(t-t_0)}},s-s_0}(q),$$

for $0 < |t_0| \ll |s_0| \ll 1$.

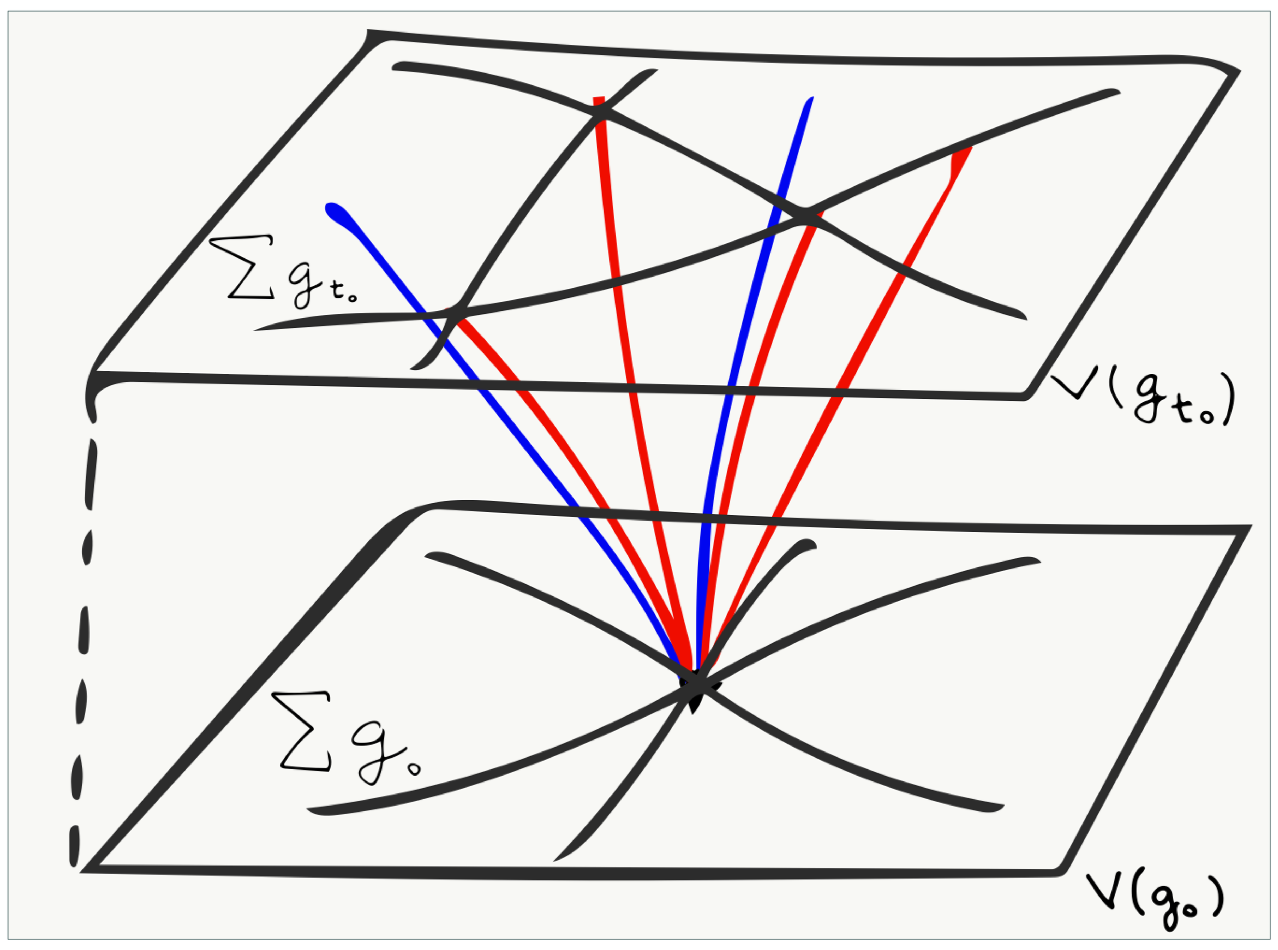

Figure 1: An IPA-deformation of a surface with one-dimensional critical locus. The red curves are components of the Lê cycle $\Lambda^1_{g,(t,s)}$ and characteristic polar cycle $\Lambda^1_{\mathbf{N}^\bullet_F,(t,s)}$. The blue curves are components of the relative polar curve $\Gamma^1_{g,t}$.

Conclusions

Rewriting the formula for $\lambda^0_{g_0,s}(\mathbf{0})$ in the form

$$\lambda^0_{g_0,s}(\mathbf{0}) + \lambda^0_{\mathbf{N}^\bullet_{f_0},s}(\mathbf{0}) = \sum_{p \in B_\epsilon \cap V(t-t_0)} \left(\lambda^0_{g_{t_0},s}(\mathbf{0}) + \lambda^0_{\mathbf{N}^\bullet_{f_{t_0}},s}(p) \right)$$

suggests a sort of "**conservation of number**" property for these quantities; is there something deeper going on here? Our **intuition** behind the generalization of Milnor's result is that

$$\begin{array}{c}\text{Change in Milnor numbers} \\ \text{from } g_0 \text{ to } g_{t_0}\end{array} = -\left(\begin{array}{l}\text{Change in char. polar mult.} \\ \text{of } \mathbf{N}^\bullet_F \text{ from } V(g_0) \text{ to } V(g_{t_0})\end{array} \right)$$

Future Directions

Question 1: Can this be extended to parameterizable hypersurface singularities of arbitrary dimension?

Question 2: Can this be extended to parameterizable LCI singularities?

Question 3: Is there really a conservation of number property at work here?

References

[1] Hepler, B. and Massey, D. Milnor Fibers inside Parameterized Hypersurfaces. *Publ. RIMS Kyoto Univ.*, 52:413–433, 2016.

[2] Massey, D. Numerical Invariants of Perverse Sheaves. *Duke Math. J.*, 73(2):307–370, 1994.